PLANE AND SPHERICAL TRIGONOMETRY

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PLANE AND SPHERICAL TRIGONOMETRY

By **PAUL R. RIDER, Ph.D.**
PROFESSOR OF MATHEMATICS
WASHINGTON UNIVERSITY

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Preface

The primary purpose of this book is to present in a sound pedagogical manner the usual course in trigonometry as offered in colleges and technical schools. Only those methods are employed which have withstood the test of many years of actual classroom use. The arrangement of topics is such as has been found desirable as a result of long experience. Even logical order has at times been sacrificed to make the material more teachable. For example, the special definitions of the trigonometric functions for acute angles are given before the more general definitions. Applications are introduced early, as it has been found that the student's interest in a subject is considerably stimulated if he can see the utility of it. Moreover, the first problems have been made simple from a numerical standpoint in order to enable him to grasp principles and to learn methods without becoming lost in a maze of computations. Formulas are developed as needed, so that there is a certain amount of purposeful alternation between theoretical and practical aspects. On the other hand, the discussion of the more difficult of the theoretical topics is postponed to the latter part of the book. Many students find it easier to solve triangles than to handle some of the analytic phases of trigonometry such as proving identities and solving equations. By solving triangles they acquire confidence, as well as a certain amount of familiarity with the relations among the functions, so that they have a greater chance of success when they tackle the more difficult portions of the subject. Too much analytic work in

the early part of the course has been found to discourage many students and to kill their interest.

A few other features of the book seem worthy of note. An effort has been made to introduce simplifications into the treatment of certain topics, notably logarithms. The use of approximate numbers in computation and the question of significant figures have been stressed. Emphasis has been placed on the orderly arrangement of computations. Sets of carefully chosen and carefully graded exercises are to be found throughout the book. Answers to the odd-numbered exercises are printed at the back, answers to the even-numbered exercises are available in pamphlet form.

The book contains a complete course in plane and spherical trigonometry as these subjects are ordinarily taught. The part on spherical trigonometry has been made rather comprehensive in view of the present interest in subjects requiring a knowledge of this branch of mathematics. The student who has mastered this part will be well equipped to pursue courses in navigation and avigation, astronomy, and other applications. If a shorter course in plane trigonometry is desired, those topics marked with a ⋆ may be omitted. A thorough course in computational trigonometry is provided by the first seven chapters. Although, as stated above, the arrangement of material is that which seemed most desirable, the separate chapters are to a large extent independent, so that the instructor who prefers a different order of presentation should have no difficulty in outlining a course to his taste.

Advice concerning some of the figures and assistance with them were kindly given by my colleagues, Professors W. H. Roever and R. W. Bockhorst, to whom I am very grateful.

My thanks are due to The Macmillan Company for making every effort to give the book a pleasing format, and for the very valuable editorial assistance which they

rendered during its preparation. The manuscript was critically read by five different advisers, and the suggestions of these advisers were given thoughtful consideration during the process of revision. The revised manuscript was then read in great detail by one of these advisers, who even worked all of the exercises. It is hoped that because of its careful preparation the book will be found both clear and teachable, as well as mathematically sound.

P. R. R.

WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI
January, 1942

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PLANE TRIGONOMETRY

★These topics may be omitted.

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*These topics may be omitted.

SPHERICAL TRIGONOMETRY

PLANE TRIGONOMETRY

CHAPTER I

Trigonometric Functions of Acute Angles

1. Trigonometry.

The word **trigonometry** is derived from the Greek and means "measurement of triangles." The subject is principally concerned with the measurement of triangles (i.e., their sides and angles), or, more specifically, with the indirect measurement of line segments and angles. For example, it is possible, by trigonometry, to measure the width of a river without crossing it, or the height of a pole or cliff without climbing to the top.

The uses of trigonometry are many. The sciences of physics, mechanics, and astronomy could hardly have developed without it; practical arts, such as engineering, find it indispensable. It is a valuable aid in the study of periodic phenomena such as the tides, or even economic data which seem to be cyclic in their nature. Various specific uses will be illustrated throughout the book, particularly in the examples and exercises.

2. Trigonometric functions of an acute angle.

Let us consider the right triangle ABC, with the right angle at C (Fig. 1). The sides opposite the angles A, B, C will be denoted by the corresponding small letters, a, b, c, respectively. Then, by taking ratios of the sides of the triangle, we define three **trigonometric functions** of the acute angle A as follows:

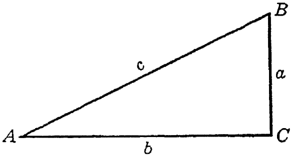


FIG. 1

sine of A (abbreviated **sin** A)

$$= \frac{\text{side opposite } A}{\text{hypotenuse}} = \frac{a}{c}, \tag{1}$$

cosine of A (abbreviated **cos** A)

$$= \frac{\text{side adjacent to } A}{\text{hypotenuse}} = \frac{b}{c}, \tag{2}$$

tangent of A (abbreviated **tan** A)

$$= \frac{\text{side opposite } A}{\text{side adjacent to } A} = \frac{a}{b}. \tag{3}$$

Thus, for example, in a right triangle in which $a = 3$, $b = 4$, $c = 5$ (see Fig. 2), we have

$$\sin A = \frac{3}{5}, \qquad \cos A = \frac{4}{5}, \qquad \tan A = \frac{3}{4}.$$

The values of these functions are completely determined by the angle A. Thus, if we had another right triangle with the same acute angle A, it would be similar to the above triangle and its sides would be in the same proportion. For example, they might all be twice as long, namely, $a = 6$, $b = 8$, $c = 10$. Then we should have $\sin A = 6/10 = 3/5$, as before, and similarly for the other functions. On the other hand, if the size of angle A were changed, the values of these functions would be changed.

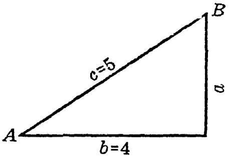


FIG. 2

Three, and only three, other ratios may also be formed from the sides of the triangle ABC. They are

cosecant of A (abbreviated **csc** A)

$$= \frac{\text{hypotenuse}}{\text{side opposite } A} = \frac{c}{a}, \tag{4}$$

secant of A (abbreviated **sec** A)

$$= \frac{\text{hypotenuse}}{\text{side adjacent to } A} = \frac{c}{b}, \tag{5}$$

cotangent of A (abbreviated **cot** A)

$$= \frac{\text{side adjacent to } A}{\text{side opposite } A} = \frac{b}{a}. \qquad (6)$$

It will be noted that these three functions are the reciprocals * of the other three, and we may write

$$\begin{aligned} \csc A &= \frac{1}{\sin A}, & \sin A &= \frac{1}{\csc A}, \\ \sec A &= \frac{1}{\cos A}, & \cos A &= \frac{1}{\sec A}, \qquad (7) \\ \cot A &= \frac{1}{\tan A}, & \tan A &= \frac{1}{\cot A}. \end{aligned}$$

NOTE. Three other functions are:

versed sine of A (abbreviated **vers** A) $= 1 - \cos A$,
coversed sine of A (abbreviated **covers** A) $= 1 - \sin A$,
haversine of A (abbreviated **hav** A) $= \frac{1}{2}(1 - \cos A)$.

They will not be used in this book.

EXERCISES I. A

Draw the right triangles whose sides have the following values, and find the six trigonometric functions of the angle A:

1. $a = 4, b = 3, c = 5.$

2. $a = 5, b = 12, c = 13.$

3. $a = 2, b = 3, c = \sqrt{13}.$

4. $a = 1, b = 1, c = \sqrt{2}.$

5. $a = 2, b = \sqrt{5}, c = 3.$

6. $a = \sqrt{2}, b = \sqrt{3}, c = \sqrt{5}.$

7. $a = 8, b = 15.$

8. $b = 21, c = 29.$

9. $a = 7, c = 25.$

10. $a = 5, b = 3.$

11. $a = 1, b = \sqrt{3}.$

12. $a = 1, b = 3.$

13. $a = 1, b = \frac{1}{3}.$

14. $a = \frac{1}{2}, b = \frac{1}{3}.$

15. A guy wire 15 feet long is fastened to a point 13 feet above the foot of a vertical pole, which stands on level ground. Find the sine of the angle that the wire makes with the horizontal.

* The reciprocal of a number is 1 divided by the number.

16. A yardstick, held vertically on a level surface, casts a shadow 1 foot 8 inches long. Find the tangent of the angle that the rays of the sun make with the horizontal.
17. A roadway rises 55 feet in a horizontal distance of $\frac{1}{2}$ mile. Find the tangent of the angle that it makes with the horizontal.
18. An airplane is descending 225 feet per 1000 feet of horizontal distance covered. What is the cosine of the angle that its path of descent makes with the horizontal?
19. One end of a foot ruler is placed against a vertical wall; the other end of the ruler reaches a point on the floor 9 inches from the base of the wall. Find the sine, cosine, and tangent of the angle that the ruler makes (a) with the wall, (b) with the floor.
20. A box is 3 inches by 4 inches by 1 foot. Find the sine of the angle that a diagonal of the box makes with its longest edge.

3. Functions of complementary angles.

By referring to the definitions of the trigonometric functions (section 2) and to Fig. 1, we see that, for the acute angle B,

$$\begin{aligned} \sin B &= \frac{b}{c}, & \csc B &= \frac{c}{b}, \\ \cos B &= \frac{a}{c}, & \sec B &= \frac{c}{a}, \\ \tan B &= \frac{b}{a}, & \cot B &= \frac{a}{b}. \end{aligned} \tag{1}$$

Comparing these formulas with formulas (1)–(6) of section 2, and making use of the fact that A and B are complementary angles (i.e., $A + B = 90°$), we have

$$\begin{aligned} \sin B &= \sin(90° - A) = \cos A, \\ \cos B &= \cos(90° - A) = \sin A, \\ \tan B &= \tan(90° - A) = \cot A, \\ \csc B &= \csc(90° - A) = \sec A, \\ \sec B &= \sec(90° - A) = \csc A, \\ \cot B &= \cot(90° - A) = \tan A. \end{aligned} \tag{2}$$

It is convenient to arrange the functions in pairs as follows: sine and cosine, tangent and cotangent, secant and cosecant. In any pair, either function may be called the **cofunction** of the other. Relations (2) may then be expressed by the single statement: *Any function of the complement of an angle is equal to the cofunction of the angle.*

EXERCISES I. B

Find the functions of angle B in exercises I. A, 1–14.

4. Finding the other functions of an acute angle when one function is given.

The following examples will illustrate how the remaining functions of an acute angle can be found if the value of one function is given.

Example 1.

Given $\sin A = \frac{5}{13}$, A acute; find the other functions of A.

Solution. Since $\sin A = \dfrac{a}{c}$, we have $\dfrac{a}{c} = \dfrac{5}{13}$. Construct a right triangle with $a = 5$ and $c = 13$ (Fig. 3). (Note that it is not necessary to take $a = 5$ and $c = 13$; we could take $a = 10$ and $c = 26$, for example, or any other numbers in the ratio of 5 to 13.)

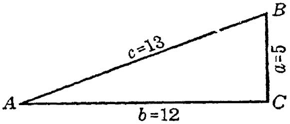


Fig. 3

Making use of the theorem of Pythagoras, that the square of the hypotenuse is equal to the sum of the squares of the sides, we have

$$b^2 = c^2 - a^2 = 169 - 25 = 144, \qquad b = 12.$$

The remaining functions of A can be read from the figure. Thus,

$$\cos A = \frac{12}{13}, \tan A = \frac{5}{12}, \csc A = \frac{13}{5}, \sec A = \frac{13}{12}, \cot A = \frac{12}{5}.$$

Example 2.

If $\tan A = 3$, what are the other functions of A, it being understood that A is acute?

SOLUTION. $$\tan A = 3 = \frac{a}{b}\cdot$$

Take $a = 3$, $b = 1$, and construct a right triangle (Fig. 4). Then,

$$c^2 = a^2 + b^2 = 9 + 1 = 10, \qquad c = \sqrt{10}.$$

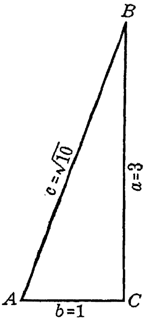


FIG. 4

$$\sin A = \frac{}{\sqrt{10}} \quad \frac{3\sqrt{10}}{10} = 0.9487,$$

$$\cos A = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} = 0.3162,$$

$$\csc A = \frac{\sqrt{10}}{3} = 1.054,$$

$$\sec A = \frac{\sqrt{10}}{1} = \sqrt{10} = 3.162$$

$$\cot A = \frac{1}{3} = 0.3333.$$

EXERCISES I. C

Find the other five functions of the acute angle A, given that

1. $\cos A = \frac{4}{5}$. **2.** $\tan A = \frac{2}{3}$. **3.** $\cot A = \frac{1}{5}$.
4. $\sin A = \frac{2}{5}$. **5.** $\sec A = \sqrt{2}$. **6.** $\csc A = \frac{41}{9}$.
7. $\sin A = \frac{1}{2}$. **8.** $\cos A = \frac{2}{3}$. **9.** $\tan A = \frac{2}{5}$.
10. $\csc A = \frac{4}{3}$. **11.** $\cot A = \frac{5}{2}$. **12.** $\sec A = \frac{5}{4}$.
13. $\sec A = 2$. **14.** $\cos A = \frac{1}{4}$. **15.** $\tan A = 0.5$.
16. $\sin A = 0.8$. **17.** $\sin A \quad \frac{\sqrt{3}}{2}$ **18.** $\cos A = \frac{\sqrt{2}}{2}\cdot$
19. $\tan A = \frac{\sqrt{3}}{3}\cdot$ **20.** $\csc A = \sqrt{2}$. **21.** $\sin A = \frac{2}{7}$.
22. $\tan A =$ **23.** $\sin A = \dfrac{2mn}{m^2 + n^2}$

24. Show that if A is an acute angle,

$$\sin^2 A + \cos^2 A = 1.$$

(The notation $\sin^2 A$ means the square of the sine of A. For example, if $\sin A = \frac{2}{3}$, then $\sin^2 A = (\frac{2}{3})^2 = \frac{4}{9}$.)

SOLUTION. $\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

$$= \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

since (see Fig. 5), by the Pythagorean theorem, $a^2 + b^2 = c^2$.

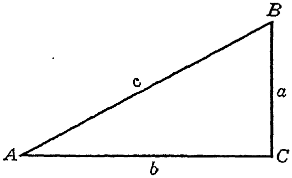


FIG. 5

Show that if A is an acute angle, then

25. $\sec^2 A = 1 + \tan^2 A.$

26. $\csc^2 A = 1 + \cot^2 A.$

27. $\cos A \tan A = \sin A.$

28. $\cot A \cos A = \csc A - \sin A.$

29. $\dfrac{1 + \sin A}{\cos A} = \dfrac{\cos A}{1 - \sin A}.$ **30.** $\dfrac{\cos^2 A}{1 - \sin A} = 1 + \sin A.$

31. $\dfrac{\sin A + \tan A}{\cot A + \csc A} = \sin A \tan A.$

32. $\dfrac{1 - 2\cos^2 A}{\sin A \cos A} = \tan A - \cot A.$

5. Functions of 45°, 60°, and 30°.

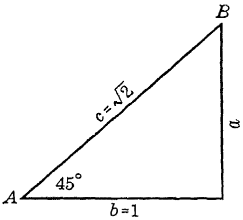


FIG. 6

To find the functions of 45° we construct an isosceles right triangle (Fig. 6). It is convenient to make each leg equal to 1, that is, $a = 1$, $b = 1$. Then,

$$c^2 = a^2 + b^2 = 1 + 1 = 2, \qquad c = \sqrt{2}.$$

From the figure we read

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.7071, \qquad \csc 45° = \sqrt{2} = 1.414,$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.7071, \qquad \sec 45° = \sqrt{2} = 1.414,$$
$$\tan 45° = 1, \qquad \cot 45° = 1.$$

The decimal values are, of course, merely approximate.

In order to find the functions of 60° we take an equilateral triangle and draw the bisector of one of the angles. (See Fig. 7.) This bisector divides the equilateral triangle into two congruent right triangles whose angles are 60° and 30°. Let us consider one of these, namely ABC. If each side of the original equilateral triangle is 2 units in length, it follows that in ABC, $c = 2$ and $b = 1$, since AC is half the base of the equilateral triangle. Then

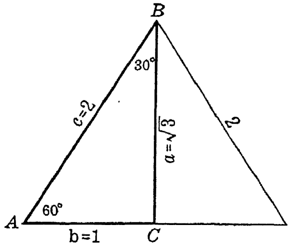


FIG. 7

$$a^2 = c^2 - b^2 = 4 - 1 = 3, \qquad a = \sqrt{3}.$$

From Fig. 7 we read

$$\sin 60° = \frac{\sqrt{3}}{2} = 0.8660, \qquad \csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.155,$$
$$\cos 60° = \frac{1}{2} = 0.5, \qquad \sec 60° = 2,$$
$$\tan 60° = \sqrt{3} = 1.732, \qquad \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.5774.$$

From the same figure, or from the relations between the functions of complementary angles, we find

$$\sin 30° = \frac{1}{2} = 0.5,$$
$$\cos 30° = \frac{\sqrt{3}}{2} = 0.8660,$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} \quad \sqrt{3} \quad 0.5774,$$

$$\csc 30^\circ = 2,$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.155,$$

$$\cot 30^\circ = \sqrt{3} = 1.732.$$

6. Tables of functions.

There are very few angles whose functions can be found by the foregoing methods of elementary geometry. It is possible, however, by other means to calculate the functions of any angle. Values of the functions have been calculated and tabulated, as for example in the table on pages 12–14, which gives the values of the sine, cosine, tangent, and cotangent of all angles from 0° to 90° for intervals of ten minutes.

To find a function of an angle less than 45° we locate the angle at the left-hand side of the table and the name of the function at the top of the column. Angles greater than 45° are located at the right-hand side of the table, and the names of their functions are located at the bottom. Opposite the angle, in the appropriate column, is found the value of the function.

For example, we find the sine of 32° 40′ to be 0.5398. Note that this is also the cosine of 57° 20′, the complement of 32° 40′. Because of the relations between the functions of an angle and the functions of its complement, the table does double duty.

EXERCISES I. D

Find, in the table on pages 12–14, the values of the following:

1. cos 28° 20′. **2.** sin 67° 30′. **3.** tan 15° 40′.
4. cot 79° 10′. **5.** sin 45° 20′. **6.** sin 0° 10′.
7. tan 0° 10′. **8.** sin 89°. **9.** tan 89° 50′.

angle	sin	tan	cot	cos	
0° 00′	.0000	.0000	—	1.0000	**90°** 00′
10	.0029	.0029	343.77	1.0000	50
20	.0058	.0058	171.89	1.0000	40
30	.0087	.0087	114.59	1.0000	30
40	.0116	.0116	85.940	.9999	20
50	.0145	.0145	68.750	.9999	10
1° 00′	.0175	.0175	57.290	.9998	**89°** 00′
10	.0204	.0204	49.104	.9998	50
20	.0233	.0233	42.964	.9997	40
30	.0262	.0262	38.188	.9997	30
40	.0291	.0291	34.368	.9996	20
50	.0320	.0320	31.242	.9995	10
2° 00′	.0349	.0349	28.636	.9994	**88°** 00′
10	.0378	.0378	26.432	.9993	50
20	.0407	.0407	24.542	.9992	40
30	.0436	.0437	22.904	.9990	30
40	.0465	.0466	21.470	.9989	20
50	.0494	.0495	20.206	.9988	10
3° 00′	.0523	.0524	19.081	.9986	**87°** 00′
10	.0552	.0553	18.075	.9985	50
20	.0581	.0582	17.169	.9983	40
30	.0610	.0612	16.350	.9981	30
40	.0640	.0641	15.605	.9980	20
50	.0669	.0670	14.924	.9978	10
4° 00′	.0698	.0699	14.301	.9976	**86°** 00′
10	.0727	.0729	13.727	.9974	50
20	.0756	.0758	13.197	.9971	40
30	.0785	.0787	12.706	.9969	30
40	.0814	.0816	12.251	.9967	20
50	.0843	.0846	11.826	.9964	10
5° 00′	.0872	.0875	11.430	.9962	**85°** 00′
10	.0901	.0904	11.059	.9959	50
20	.0929	.0934	10.712	.9957	40
30	.0958	.0963	10.385	.9954	30
40	.0987	.0992	10.078	.9951	20
50	.1016	.1022	9.7882	.9948	10
6° 00′	.1045	.1051	9.5144	.9945	**84°** 00′
10	.1074	.1080	9.2553	.9942	50
20	.1103	.1110	9.0098	.9939	40
30	.1132	.1139	8.7769	.9936	30
40	.1161	.1169	8.5555	.9932	20
50	.1190	.1198	8.3450	.9929	10
7° 00′	.1219	.1228	8.1443	.9925	**83°** 00′
10	.1248	.1257	7.9530	.9922	50
20	.1276	.1287	7.7704	.9918	40
30	.1305	.1317	7.5958	.9914	30
40	.1334	.1346	7.4287	.9911	20
50	.1363	.1376	7.2687	.9907	10
8° 00′	.1392	.1405	7.1154	.9903	**82°** 00′
10	.1421	.1435	6.9682	.9899	50
20	.1449	.1465	6.8269	.9894	40
30	.1478	.1495	6.6912	.9890	30
40	.1507	.1524	6.5606	.9886	20
50	.1536	.1554	6.4348	.9881	10
9° 00′	.1564	.1584	6.3138	.9877	**81°** 00′
	cos	cot	tan	sin	angle

angle	sin	tan	cot	cos	
9° 00′	.1564	.1584	6.3138	.9877	**81°** 00′
10	.1593	.1614	6.1970	.9872	50
20	.1622	.1644	6.0844	.9868	40
30	.1650	.1673	5.9758	.9863	30
40	.1679	.1703	5.8708	.9858	20
50	.1708	.1733	5.7694	.9853	10
10° 00′	.1736	.1763	5.6713	.9848	**80°** 00′
10	.1765	.1793	5.5764	.9843	50
20	.1794	.1823	5.4845	.9838	40
30	.1822	.1853	5.3955	.9833	30
40	.1851	.1883	5.3093	.9827	20
50	.1880	.1914	5.2257	.9822	10
11° 00′	.1908	.1944	5.1446	.9816	**79°** 00′
10	.1937	.1974	5.0658	.9811	50
20	.1965	.2004	4.9894	.9805	40
30	.1994	.2035	4.9152	.9799	30
40	.2022	.2065	4.8430	.9793	20
50	.2051	.2095	4.7729	.9787	10
12° 00′	.2079	.2126	4.7046	.9781	**78°** 00′
10	.2108	.2156	4.6382	.9775	50
20	.2136	.2186	4.5736	.9769	40
30	.2164	.2217	4.5107	.9763	30
40	.2193	.2247	4.4494	.9757	20
50	.2221	.2278	4.3897	.9750	10
13° 00′	.2250	.2309	4.3315	.9744	**77°** 00′
10	.2278	.2339	4.2747	.9737	50
20	.2306	.2370	4.2193	.9730	40
30	.2334	.2401	4.1653	.9724	30
40	.2363	.2432	4.1126	.9717	20
50	.2391	.2462	4.0611	.9710	10
14° 00′	.2419	.2493	4.0108	.9703	**76°** 00′
10	.2447	.2524	3.9617	.9696	50
20	.2476	.2555	3.9136	.9689	40
30	.2504	.2586	3.8667	.9681	30
40	.2532	.2617	3.8208	.9674	20
50	.2560	.2648	3.7760	.9667	10
15° 00′	.2588	.2679	3.7321	.9659	**75°** 00′
10	.2616	.2711	3.6891	.9652	50
20	.2644	.2742	3.6470	.9644	40
30	.2672	.2773	3.6059	.9636	30
40	.2700	.2805	3.5656	.9628	20
50	.2728	.2836	3.5261	.9621	10
16° 00′	.2756	.2867	3.4874	.9613	**74°** 00′
10	.2784	.2899	3.4495	.9605	50
20	.2812	.2931	3.4124	.9596	40
30	.2840	.2962	3.3759	.9588	30
40	.2868	.2994	3.3402	.9580	20
50	.2896	.3026	3.3052	.9572	10
17° 00′	.2924	.3057	3.2709	.9563	**73°** 00′
10	.2952	.3089	3.2371	.9555	50
20	.2979	.3121	3.2041	.9546	40
30	.3007	.3153	3.1716	.9537	30
40	.3035	.3185	3.1397	.9528	20
50	.3062	.3217	3.1084	.9520	10
18° 00′	.3090	.3249	3.0777	.9511	**72°** 00′
	cos	cot	tan	sin	angle

angle	sin	tan	cot	cos	
18° 00′	.3090	.3249	3.0777	.9511	**72°** 00′
10	.3118	.3281	3.0475	.9502	50
20	.3145	.3314	3.0178	.9492	40
30	.3173	.3346	2.9887	.9483	30
40	.3201	.3378	2.9600	.9474	20
50	.3228	.3411	2.9319	.9465	10
19° 00′	.3256	.3443	2.9042	.9455	**71°** 00′
10	.3283	.3476	2.8770	.9446	50
20	.3311	.3508	2.8502	.9436	40
30	.3338	.3541	2.8239	.9426	30
40	.3365	.3574	2.7980	.9417	20
50	.3393	.3607	2.7725	.9407	10
20° 00′	.3420	.3640	2.7475	.9397	**70°** 00′
10	.3448	.3673	2.7228	.9387	50
20	.3475	.3706	2.6985	.9377	40
30	.3502	.3739	2.6746	.9367	30
40	.3529	.3772	2.6511	.9356	20
50	.3557	.3805	2.6279	.9346	10
21° 00′	.3584	.3839	2.6051	.9336	**69°** 00′
10	.3611	.3872	2.5826	.9325	50
20	.3638	.3906	2.5605	.9315	40
30	.3665	.3939	2.5386	.9304	30
40	.3692	.3973	2.5172	.9293	20
50	.3719	.4006	2.4960	.9283	10
22° 00′	.3746	.4040	2.4751	.9272	**68°** 00′
10	.3773	.4074	2.4545	.9261	50
20	.3800	.4108	2.4342	.9250	40
30	.3827	.4142	2.4142	.9239	30
40	.3854	.4176	2.3945	.9228	20
50	.3881	.4210	2.3750	.9216	10
23° 00′	.3907	.4245	2.3559	.9205	**67°** 00′
10	.3934	.4279	2.3369	.9194	50
20	.3961	.4314	2.3183	.9182	40
30	.3987	.4348	2.2998	.9171	30
40	.4014	.4383	2.2817	.9159	20
50	.4041	.4417	2.2637	.9147	10
24° 00′	.4067	.4452	2.2460	.9135	**66°** 00′
10	.4094	.4487	2.2286	.9124	50
20	.4120	.4522	2.2113	.9112	40
30	.4147	.4557	2.1943	.9100	30
40	.4173	.4592	2.1775	.9088	20
50	.4200	.4628	2.1609	.9075	10
25° 00′	.4226	.4663	2.1445	.9063	**65°** 00′
10	.4253	.4699	2.1283	.9051	50
20	.4279	.4734	2.1123	.9038	40
30	.4305	.4770	2.0965	.9026	30
40	.4331	.4806	2.0809	.9013	20
50	.4358	.4841	2.0655	.9001	10
26° 00′	.4384	.4877	2.0503	.8988	**64°** 00′
10	.4410	.4913	2.0353	.8975	50
20	.4436	.4950	2.0204	.8962	40
30	.4462	.4986	2.0057	.8949	30
40	.4488	.5022	1.9912	.8936	20
50	.4514	.5059	1.9768	.8923	10
27° 00′	.4540	.5095	1.9626	.8910	**63°** 00′
	cos	cot	tan	sin	angle

angle	sin	tan	cot	cos	
27° 00′	.4540	.5095	1.9626	.8910	**63°** 00′
10	.4566	.5132	1.9486	.8897	50
20	.4592	.5169	1.9347	.8884	40
30	.4617	.5206	1.9210	.8870	30
40	.4643	.5243	1.9074	.8857	20
50	.4669	.5280	1.8940	.8843	10
28° 00′	.4695	.5317	1.8807	.8829	**62°** 00′
10	.4720	.5354	1.8676	.8816	50
20	.4746	.5392	1.8546	.8802	40
30	.4772	.5430	1.8418	.8788	30
40	.4797	.5467	1.8291	.8774	20
50	.4823	.5505	1.8165	.8760	10
29° 00′	.4848	.5543	1.8040	.8746	**61°** 00′
10	.4874	.5581	1.7917	.8732	50
20	.4899	.5619	1.7796	.8718	40
30	.4924	.5658	1.7675	.8704	30
40	.4950	.5696	1.7556	.8689	20
50	.4975	.5735	1.7437	.8675	10
30° 00′	.5000	.5774	1.7321	.8660	**60°** 00′
10	.5025	.5812	1.7205	.8646	50
20	.5050	.5851	1.7090	.8631	40
30	.5075	.5890	1.6977	.8616	30
40	.5100	.5930	1.6864	.8601	20
50	.5125	.5969	1.6753	.8587	10
31° 00′	.5150	.6009	1.6643	.8572	**59°** 00′
10	.5175	.6048	1.6534	.8557	50
20	.5200	.6088	1.6426	.8542	40
30	.5225	.6128	1.6319	.8526	30
40	.5250	.6168	1.6212	.8511	20
50	.5275	.6208	1.6107	.8496	10
32° 00′	.5299	.6249	1.6003	.8480	**58°** 00′
10	.5324	.6289	1.5900	.8465	50
20	.5348	.6330	1.5798	.8450	40
30	.5373	.6371	1.5697	.8434	30
40	.5398	.6412	1.5597	.8418	20
50	.5422	.6453	1.5497	.8403	10
33° 00′	.5446	.6494	1.5399	.8387	**57°** 00′
10	.5471	.6536	1.5301	.8371	50
20	.5495	.6577	1.5204	.8355	40
30	.5519	.6619	1.5108	.8339	30
40	.5544	.6661	1.5013	.8323	20
50	.5568	.6703	1.4919	.8307	10
34° 00′	.5592	.6745	1.4826	.8290	**56°** 00′
10	.5616	.6787	1.4733	.8274	50
20	.5640	.6830	1.4641	.8258	40
30	.5664	.6873	1.4550	.8241	30
40	.5688	.6916	1.4460	.8225	20
50	.5712	.6959	1.4370	.8208	10
35° 00′	.5736	.7002	1.4281	.8192	**55°** 00′
10	.5760	.7046	1.4193	.8175	50
20	.5783	.7089	1.4106	.8158	40
30	.5807	.7133	1.4019	.8141	30
40	.5831	.7177	1.3934	.8124	20
50	.5854	.7221	1.3848	.8107	10
36° 00′	.5878	.7265	1.3764	.8090	**54°** 00′
	cos	cot	tan	sin	angle

Trigonometric Functions

angle	sin	tan	cot	cos	
36° 00′	.5878	.7265	1.3764	.8090	**54°** 00′
10	.5901	.7310	1.3680	.8073	50
20	.5925	.7355	1.3597	.8056	40
30	.5948	.7400	1.3514	.8039	30
40	.5972	.7445	1.3432	.8021	20
50	.5995	.7490	1.3351	.8004	10
37° 00′	.6018	.7536	1.3270	.7986	**53°** 00′
10	.6041	.7581	1.3190	.7969	50
20	.6065	.7627	1.3111	.7951	40
30	.6088	.7673	1.3032	.7934	30
40	.6111	.7720	1.2954	.7916	20
50	.6134	.7766	1.2876	.7898	10
38° 00′	.6157	.7813	1.2799	.7880	**52°** 00′
10	.6180	.7860	1.2723	.7862	50
20	.6202	.7907	1.2647	.7844	40
30	.6225	.7954	1.2572	.7826	30
40	.6248	.8002	1.2497	.7808	20
50	.6271	.8050	1.2423	.7790	10
39° 00′	.6293	.8098	1.2349	.7771	**51°** 00′
10	.6316	.8146	1.2276	.7753	50
20	.6338	.8195	1.2203	.7735	40
30	.6361	.8243	1.2131	.7716	30
40	.6383	.8292	1.2059	.7698	20
50	.6406	.8342	1.1988	.7679	10
40° 00′	.6428	.8391	1.1918	.7660	**50°** 00′
10	.6450	.8441	1.1847	.7642	50
20	.6472	.8491	1.1778	.7623	40
30	.6494	.8541	1.1708	.7604	30
40	.6517	.8591	1.1640	.7585	20
50	.6539	.8642	1.1571	.7566	10
41° 00′	.6561	.8693	1.1504	.7547	**49°** 00′
10	.6583	.8744	1.1436	.7528	50
20	.6604	.8796	1.1369	.7509	40
30	.6626	.8847	1.1303	.7490	30
40	.6648	.8899	1.1237	.7470	20
50	.6670	.8952	1.1171	.7451	10
42° 00′	.6691	.9004	1.1106	.7431	**48°** 00′
10	.6713	.9057	1.1041	.7412	50
20	.6734	.9110	1.0977	.7392	40
30	.6756	.9163	1.0913	.7373	30
40	.6777	.9217	1.0850	.7353	20
50	.6799	.9271	1.0786	.7333	10
43° 00′	.6820	.9325	1.0724	.7314	**47°** 00′
10	.6841	.9380	1.0661	.7294	50
20	.6862	.9435	1.0599	.7274	40
30	.6884	.9490	1.0538	.7254	30
40	.6905	.9545	1.0477	.7234	20
50	.6926	.9601	1.0416	.7214	10
44° 00′	.6947	.9657	1.0355	.7193	**46°** 00′
10	.6967	.9713	1.0295	.7173	50
20	.6988	.9770	1.0235	.7153	40
30	.7009	.9827	1.0176	.7133	30
40	.7030	.9884	1.0117	.7112	20
50	.7050	.9942	1.0058	.7092	10
45° 00′	.7071	1.0000	1.0000	.7071	**45°** 00′
	cos	cot	tan	sin	angle

Find the value of the acute angle A, given that

10. $\sin A = 0.0727$. **11.** $\cos A = 0.8021$. **12.** $\tan A = 2.3183$.
13. $\cot A = 3.2709$. **14.** $\sin A = 0.6202$. **15.** $\cos A = 0.3665$.
16. $\tan A = 0.9601$. **17.** $\cot A = 6.8269$. **18.** $\sin 2A = 0.1994$.
19. $2 \sin A = 1.9500$. **20.** $\sin(A + 30°) = 0.6180$.
21. $\tan(2A - 30°) = 0.3249$. **22.** $2 \cos(\frac{1}{2}A + 10°) = 0.6786$.

23. Find the value of $\sin 20° + \sin 30°$. Is this equal to $\sin 50°$?

CHAPTER II

Solution of Triangles

7. Solution of right triangles.

The use of tables of the trigonometric functions will be illustrated by some examples.

Example 1.

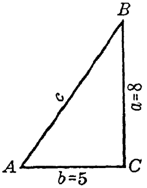


FIG. 8

A vertical pole 8 feet tall casts a shadow 5 feet long on level ground. Find the angle which the rays of the sun make with the horizontal.

SOLUTION. In Fig. 8, a represents the height of the pole, b represents the length of the shadow, A is the angle to be found. We have

$$\tan A = \frac{a}{b} = \frac{8}{5} = 1.6.$$

From the table on pages 12–14 we find $A = 58°$ (to the nearest $10'$).

Example 2.

A surveyor wishes to measure the distance across a stream. He sets up his transit at a point C on the bank of the stream, and sights on a point B on the other bank directly opposite him. Then he turns the transit through a right angle, and measures off a distance of 100 feet to a point A. He moves the transit to A, and measures the angle CAB, which he finds to be 50°. How wide is the stream?

SOLUTION. The conditions of the problem are illustrated in Fig. 9. To find a, the distance across the stream, we proceed as follows:

$$\frac{a}{b} = \tan A, \qquad a = b \tan A = 100 \tan 50°$$

From the table on pages 12–14 we find $\tan 50° = 1.1918$. Thus,

$$a = 100 \times 1.1918 = 119.2 \text{ ft.}$$

A triangle is composed of six **parts**, the three sides and the three angles. To **solve** a triangle is to find the unknown parts from the parts that are given. In the case of a right triangle this can always be done if we have given (besides the right angle) two parts, at least one of which is a side.

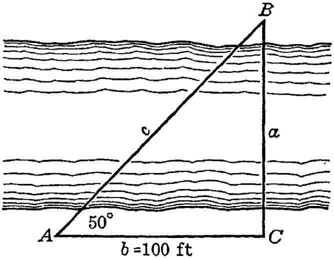


FIG. 9

In problems involving a right triangle ABC, it will ordinarily be understood that the right angle is at C.

In solving right triangles we make use of four of the definitions, namely,

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b}, \qquad \cot A = \frac{b}{a},$$

and of the Pythagorean relation,

$$a^2 + b^2 = c^2.$$

(We seldom use the secant or cosecant, since tables of these functions are not so generally available.) Of course we sometimes find it convenient to use the relation

$$A + B = 90°,$$

and the fact that the functions of B are equal respectively to the corresponding cofunctions of A.

From the foregoing relations we select one which contains the two given, or known, parts and the part which we wish to find.

Example 3.

Solve the right triangle ABC in which $c = 25$, $A = 32° 10'$.

SOLUTION. To find a we use the definition $a/c = \sin A$, which contains the known parts c and A. We get

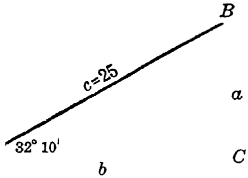


FIG. 10

$$a = c \sin A = 25 \sin 32° 10'$$
$$= 25 \times 0.5324 = 13.3.$$

To find b we use $b/c = \cos A$, from which we get

$$b = c \cos A = 25 \cos 32° 10'$$
$$= 25 \times 0.8465 = 21.2.$$

$$\begin{aligned} 90° &= 89° 60' \\ A &= 32° 10' \\ \hline B &= 57° 50' \end{aligned}$$

Example 4.

Given $a = 27.2$, $b = 10.6$; find A, B, c.

SOLUTION.

$$\tan A = \frac{a}{b} = \frac{27.2}{10.6} = 2.5660, \qquad A = 68° 40'.$$

The value 2.5660 is not to be found in the table on pages 12–14. The value closest to this is 2.5605, which is the tangent of 68° 40'. Consequently, as an approximation, we take

$$A = 68° 40'.$$

In a later section we shall learn how to find a more accurate value for an angle when the given function is between two consecutive values in the table.

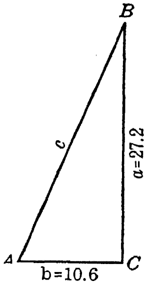


FIG. 11

$$B = 90° - A = 21° 20'.$$

$$\frac{a}{c} = \sin A, \qquad c \sin A = a,$$

$$c = \frac{\quad}{\sin A} = \frac{27.2}{\sin 68° 40'} = \frac{27.2}{0.9315} = 29.2.$$

We could also find c by using the relation $c^2 = a^2 + b^2$, obtaining values from a table of squares, such as is to be found in Table VI of the Macmillan Logarithmic and Trigonometric Tables. Thus,

$$c^2 = (27.2)^2 + (10.6)^2 = 739.84 + 112.36 = 852.20.$$

From Table VI, just referred to, we find

$$c = 29.2.$$

It is recommended that all answers be checked by obtaining the solutions in two different ways.

It is also recommended that a drawing be made to scale. From such a drawing it is possible to make at least a rough check of the results.

EXERCISES II. A

In solving the following exercises, use the nearest values that are to be found in the tables.

Solve the following triangles, in which $C = 90°$.

1. $A = 35°, c = 5.$

2. $a = 6, c = 14.$

3. $A = 37°, b = 53.$

4. $B = 56°, c = 84.$

5. $a = 23, b = 17.$

6. $a = 18.5, c = 37.2.$

7. $B = 17° 30', b = 92.4.$

8. $A = 57° 20', c = 0.0286.$

9. $a = 0.257, b = 0.856.$

10. $b = 189, A = 13° 50'.$

11. A wire is stretched from the top of a vertical pole standing on level ground. The wire reaches to a point on the ground 10 feet from the foot of the pole, and makes an angle of 75° with the horizontal. Find the height of the pole and the length of the wire.

12. A flagpole broken over by the wind forms a right triangle with the ground. If the angle which the broken part makes with the ground is 50°, and the distance from the tip of the pole to the foot is 55 feet, how tall was the pole?

13. A ladder 36 feet long rests against a wall, its foot being at a horizontal distance of 25 feet from the base of the wall. What angle does the ladder make with the ground?

14. If a ladder 40 feet long is placed so as to reach a window

30 feet high, what angle does it make with the level ground, and how far is its foot from the base of the building?

15. A ladder 42 feet long is placed so that it will reach a window 30 feet high on one side of a street; if it is turned over, its foot being held in position, it will reach a window 25 feet high on the other side of the street. How wide is the street from building to building?

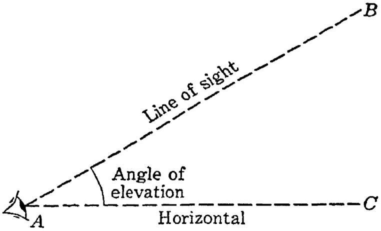


FIG. 12

16. A person on a ship sailing due south at the rate of 15 miles an hour observes a lighthouse due west at 3 p.m. At 5 p.m. the lighthouse is 52° west of north. How far from the lighthouse was the ship at (a) 3 p.m.? (b) 5 p.m.? (c) 4 p.m.?

The **angle of elevation** of an object which is above the eye of an observer is the angle which the line of sight to the object makes with the horizontal. If the object is below the eye of the observer, the angle which the line of sight makes with the horizontal is called the **angle of depression** of the object.

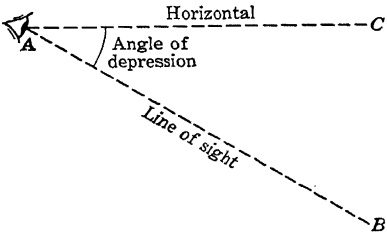


FIG. 13

17. From the top of a cliff 250 feet high the angle of depression of a boat is 10°. How far out is the boat from the foot of the cliff?

18. From a window 30 feet above the level ground, a building 100 feet high, and at a distance of 200 feet, is observed. Find the angle of elevation of the top of the building and the angle of depression of its base.

19. At a point 160 feet from a building, and in a horizontal line with its base, the angle of elevation of the top of the building is 37°. How high is the building?

8. Interpolation.

When an angle such as 18° 47′ cannot be found in the margin of the table on pages 12–14, we can approximate more closely the values of its functions by a process known as **interpolation by proportional parts.** This will be illustrated by means of examples.

Example 1.

Find sin 18° 47′.

SOLUTION. The angle 18° 47′ is between 18° 40′ and 18° 50′. Its sine is between the sines of these two angles. We write the problem in the following form, in which the differences in the angles are shown at the left, and the differences in the values of the function are shown at the right.

$$10' \left\{ \begin{array}{l} \qquad \sin 18^\circ\, 50' = .3228 \\ 7' \left\{ \begin{array}{l} \sin 18^\circ\, 47' = \quad ? \\ \sin 18^\circ\, 40' = .3201 \end{array} \right\} x \end{array} \right\} .0027$$

Although it is only approximately true, we assume that changes in the function are proportional to changes in the angle. With this assumption, we have

$$\frac{x}{0.0027} = \frac{7}{10} = 0.7, \qquad x = 0.7 \times 0.0027 = 0.00189.$$

We cut this down to four places, since we are dealing with a four-place table, and write $x = 0.0019$. Then,

$$\sin 18^\circ\, 47' = 0.3201 + 0.0019 = 0.3220.$$

This value is correct to four places, as may be verified by consulting more extensive tables.

Example 2.

Find cos 18° 47′.

SOLUTION. The same form of arrangement is used as in example 1. However, it will be noted that the smaller angle has the larger cosine, and to facilitate the subtraction of the functions we

write it above. The quantity x is used, as in example 1, to represent the unknown difference between the function of the smaller angle (not the smaller function) and the function to be found.

$$10' \left\{ \begin{array}{l} \cos 18^\circ 40' = .9474 \\ \cos 18^\circ 47' = \quad ? \\ \cos 18^\circ 50' = .9465 \end{array} \right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} x \\ \\ \end{array} \quad .0009$$

$$\frac{x}{0.0009} = \frac{7}{10} = 0.7, \qquad x = 0.7 \times 0.0009 = 0.00063.$$

Noting that the function decreases as the angle increases, we have

$$\cos 18^\circ 47' = 0.9474 - 0.0006 = 0.9468.$$

If more extensive tables are used, it will be found that the value correct to four places is actually 0.9467.

Likewise, when a function cannot be found exactly in the table, we use inverse interpolation to find the corresponding angle more accurately.

Example 3.

Given $\tan A = 1.1948$; find A.

SOLUTION. The function lies between 1.1918 (corresponding to $50^\circ 00'$) and 1.1988 (corresponding to $50^\circ 10'$).

$$10' \quad \begin{array}{l} \tan 50^\circ 10' = 1.1988 \\ x \left\{ \begin{array}{ll} \tan A & = 1.1948 \\ \tan 50^\circ 00' & = 1.1918 \end{array} \right\} .0030 \end{array} \quad .0070$$

$$\frac{x}{10'} = \frac{0.0030}{0.0070} = 0.4, \qquad x = 4'.$$

$$A = 50^\circ 4'.$$

Example 4.

Given $\cos A = 0.7034$; find A.

SOLUTION. The function lies between 0.7030 (corresponding to $45^\circ 20'$) and 0.7050 (corresponding to $45^\circ 10'$). We write the functions with the largest at the top to facilitate the subtraction.

The quantity x is used to represent the difference between the smaller of the two angles taken from the table and the angle to be found; x will then be the amount to be added to the smaller angle.

$$10' \quad x\left\{\begin{array}{l} \cos 45^\circ\, 10' \\ \cos A \end{array}\right. \begin{array}{l} = .7050 \\ = .7034 \end{array}\!\left.\vphantom{\begin{array}{l} a \\ b \end{array}}\right\} .0016 \qquad .0020$$
$$\cos 45^\circ\, 20' = .7030$$

$$\frac{x}{10'} - \frac{0.0016}{0.0020} = 0.8, \qquad x = 8'.$$
$$A = 45^\circ\, 18'.$$

The process of interpolation can be used on any table provided the values are sufficiently close together. For example, it can be used on a table of squares or a table of square roots.

EXERCISES II. B

Find, by interpolation in the table on pages 12–14, the following functions:

1. $\sin 31^\circ\, 14'$. **2.** $\tan 18^\circ\, 6'$. **3.** $\cos 27^\circ\, 18'$.
4. $\cos 39^\circ\, 42'$. **5.** $\sin 55^\circ\, 5'$. **6.** $\cot 43^\circ\, 18'$.
7. $\tan 19^\circ\, 26'$. **8.** $\sin 27^\circ\, 24'$. **9.** $\cos 45^\circ\, 34'$.
10. $\sin 0^\circ\, 3'$. **11.** $\cot 89^\circ\, 51'$. **12.** $\sin 88^\circ\, 22'$.
13. $\tan 88^\circ\, 51'$. **14.** $\cos 74^\circ\, 32'$. **15.** $\cot 65^\circ\, 17'$.

Find angle A by interpolation in the table on pages 12–14, given that

16. $\sin A = 0.4827$. **17.** $\tan A = 0.3899$. **18.** $\cos A = 0.8643$.
19. $\cot A = 2.5626$. **20.** $\tan A = 1.3900$. **21.** $\sin A = 0.3290$.
22. $\sin A = 0.8026$. **23.** $\cos A = 0.3785$. **24.** $\cot A = 0.3785$.
25. $\sin A = 0.0130$. **26.** $\tan A = 0.0130$. **27.** $\sin A = 0.1060$.
28. $\tan A = 0.1060$. **29.** $\cos A = 0.9800$. **30.** $\cot A = 2.0000$.

Solve the following triangles, in which $C = 90^\circ$

31. $a = 6.84$, $c = 20$. **32.** $a = 23$, $b = 17$.
33. $A = 57^\circ\, 12'$, $c = 0.0286$. **34.** $B = 17^\circ\, 26'$, $b = 92.37$.
35. $a = 18.5$, $c = 37.2$. **36.** $A = 32^\circ\, 24'$, $b = 9.46$.
37. $A = 19^\circ\, 44'$, $a = 22.8$. **38.** $b = 15.4$, $c = 20.2$.

39. $A = 45° 2'$, $b = 8.22$. **40.** $B = 15° 53'$, $a = 189$.

41. $a = 0.236$, $c = 1.84$. **42.** $a = 17.6$, $b = 16.7$.

43. $A = 11° 1'$, $c = 101.6$. **44.** $A = 78° 15'$, $b = 32.22$.

45. $a = 12.34$, $c = 100.3$. **46.** $a = 12.34$, $b = 100.3$.

47. A rectangle is 87 feet by 136 feet. Find the length of the diagonal and the angles that it makes with the sides.

48. A surveyor wishes to find the width of a stream without crossing it. He measures a line CB along the bank, C being directly opposite a point A on the farther bank (i.e., angle $ACB = 90°$). The line CB is measured to be 98.25 feet, and the angle ABC to be $55° 56'$. How wide is the stream?

49. Find the height of a vertical pole which casts a shadow 67 feet long on the level ground when the altitude of the sun is $50° 22'$ (i.e., the rays of the sun make an angle of $50° 22'$ with the horizontal).

50. Find the inclination, or angle of ascent, of a road having a $2\frac{1}{2}$ per cent grade (i.e., there is a vertical rise of $2\frac{1}{2}$ feet in a horizontal distance of 100 feet).

51. To measure the height of a building, a surveyor sets up his transit at a distance of 112.2 feet from the building. He finds the angle of elevation of the top of the building to be $48° 17'$. If the telescope of the transit is 5 feet above the base of the building, how high is the building?

52. From the top of a tower 63.2 feet high, the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side of the tower, are $31° 16'$ and $46° 28'$ respectively. Find the distance between the two objects.

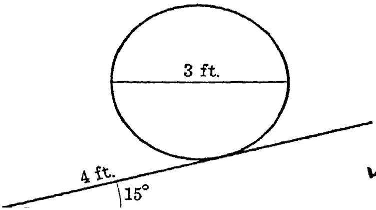


FIG. 14

✓**53** A wheel, 3 feet in diameter, rolls up an incline of 15°. When the point of contact of the wheel with the incline is 4 feet from the base of the incline, what is the height of the center of the wheel above the base of the incline?

54. A roof 20 by 30 feet, the latter being the horizontal dimension,

is inclined at an angle of 30° to the horizontal. Find the angle that a diagonal of the roof makes with the horizontal.

55. A wall extending east and west is 6 feet high. The sun has an altitude of 49° 32′ (see exercise 49) and is 47° 20′ east of south. Find the width of the shadow of the wall on level ground.

56. A 30-foot flagstaff is fixed in the center of a circular tower 40 feet in diameter. From a point in the same horizontal plane as the foot of the tower the angles of elevation of the top of the flagstaff and the top of the tower are found to be 36° and 30° respectively. Find the height of the tower.

57. If, in the preceding exercise, the flagstaff is fixed on the edge of the tower, what is the height of the tower?

58. It is required to measure the height of a tower, CB (Fig. 15), which is inaccessible. From a point A, in the same horizontal plane with the base C, a right angle CAD is turned, and a horizontal line AD, 150 feet in length, is measured. At A the angle of elevation of the top of the tower is 32°, at D the angle of elevation is 28°. Find the height of the tower.

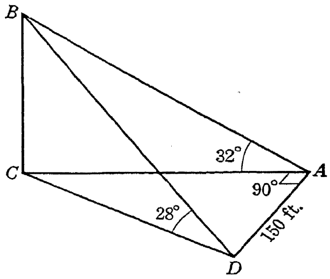


Fig. 15

✓

59. A football player stands at a distance c behind the middle of the goal. He sees the angle of elevation of the nearer crossbar to be u and that of the farther one to be v. Show that the distance between the goals is $c(\tan u \cot v - 1)$.

60. Two points in line with a tower, and in the same horizontal plane with its base, are 160 feet apart. From the point nearer the tower the angle of elevation of the top of the tower is A, from the other point the angle of elevation is B. If $\sin A = 3/5$ and $\cos B = 12/13$, what is the height of the tower?

★9. Components.

The trigonometric functions have direct application in physics and mechanics. A displacement (change of posi-

★ Topics marked with this symbol may be omitted in a short course.

tion), velocity, force, or any other quantity having both magnitude and direction, can be represented by a line having a certain length and a certain direction.

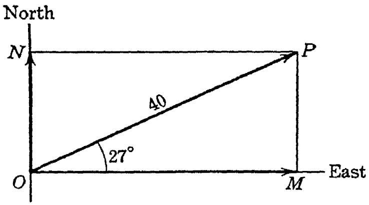


FIG. 16

For example, suppose that an automobile is traveling at the rate of 40 miles an hour along a straight road which makes an angle of 27° to the north of east. Its velocity can be represented by a line OP, 40 units long, extending in the direction shown in Fig. 16. Let M be the projection of P upon an east-west line (that is, the foot of the perpendicular from P to such a line), and let N be its projection on a north-south line. Then,

$$OM = OP \cos 27° = 40 \times 0.8910 = 35.64,$$
$$ON = OP \sin 27° = 40 \times 0.4540 = 18.16.$$

At the end of an hour the automobile will be 35.64 miles east, and 18.16 miles north, of its position at the beginning of the hour. Thus, we may think of its velocity as being composed of an easterly velocity of 35.64 miles an hour and a northerly velocity of 18.16 miles an hour. The projections OM and ON represent the **components** of the velocity represented by OP. We say that OP is **resolved** into its components OM and ON. Conversely, we say that OP is the **resultant** of OM and ON.

Example 1.

A boat, which can travel at the rate of 4 miles an hour in still water, is pointed directly across a stream having a current of 3 miles an hour. What will be the actual speed of the boat, and in what direction will the boat go?

SOLUTION. In still water the boat would go out at right angles to the bank at the rate of 4 miles an hour. But the current carries

it downstream 3 units for every 4 units that it goes across. In Fig. 17, OM represents the velocity of the current, and ON represents the velocity that the boat would have if there were no current. The actual velocity of the boat will be represented by OP. The magnitude of OP is $\sqrt{3^2 + 4^2} = 5$. If A is the angle that OP makes with the bank, then we have $\tan A = \frac{4}{3} = 1.3333$, and $A = 53°$ approximately. That is, the boat will travel at a speed of 5 miles an hour in a direction making an angle of about 53° with the bank.

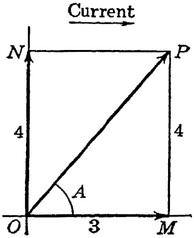


FIG. 17

Example 2.

How must the boat of the preceding example be pointed in order to go straight across the stream?

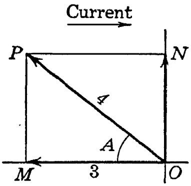


FIG. 18

SOLUTION. The boat must be pointed so that its velocity of 4 miles an hour will have a component parallel to the bank which will exactly offset the effect of the current. That is, it must have an upstream component of 3 miles an hour. From Fig. 18 we see that $\cos A = \frac{3}{4} = 0.75$, and $A = 41.5°$ approximately. Thus, to go straight across the stream, the boat should be pointed at an angle of 41.5° with the upstream direction.

Example 3.

Two forces of 100 pounds and 80 pounds respectively act on a weight as shown in Fig. 19. What will be their horizontal effect, and what will be their vertical or lifting effect?

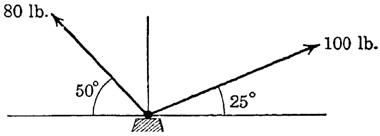


FIG. 19

SOLUTION. The horizontal component of the 100-lb. force is $100 \cos 25° = 90.63$ lb. to the right. The horizontal component of the 80-lb. force is $80 \cos 50° = 51.42$ lb. to the left. Thus, the total horizontal force tending to move the weight to the right is

$$90.63 - 51.42 = 39.21 \text{ lb.}$$

The total lifting force is

$$100 \sin 25° + 80 \sin 50° = 42.26 + 61.28 = 103.54 \text{ lb.}$$

Example 4.

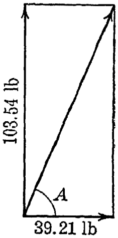


Fig. 20

Find the magnitude and the direction of the resultant force (the single force that is equivalent to the two given forces) in example 3.

Solution. The components of the resultant are 39.21 lb. to the right, and 103.54 lb. upward. The resultant force is

$$\sqrt{(39.21)^2 + (103.54)^2} = 110.7 \text{ lb.}$$

If A is the angle that the resultant makes with the horizontal,

$$\tan A = \frac{103.54}{39.21} = 2.641, \qquad A = 69° \, 15' \text{ (to nearest } 5').$$

That is, a single force of 110.7 lb., acting at an angle of 69° 15′ with the horizontal and toward the right, will have the same effect as the two given forces.

EXERCISES II. C

1. The westward and southward components of the velocity of a ship are 6.7 knots and 12.5 knots respectively. (See exercise 7.) Find the speed of the ship and the direction in which it is sailing.
2. A force of 150 pounds is acting at an angle of 55° with the horizontal. Find its horizontal and vertical components.
3. A balloon is rising at the rate of 10 feet a second and is being carried horizontally by a wind which has a velocity of 15 miles an hour. Find its actual velocity and the angle that its path makes with the vertical.
4. A boat is being rowed north at the rate of 5 miles an hour, and the tide carries it west at the rate of 3 miles an hour. Find the actual speed of the boat and the direction of its path.
5. A river flows at the rate of 1.5 miles an hour. (a) In what direction must a man swim in order to go straight across, if his

rate of swimming in still water is 2.5 miles an hour? (b) How long will it take him to cross if the river is $\frac{1}{2}$ mile wide?

6. A barge is being towed north at the rate of 15 miles an hour. A man walks across the deck, from west to east, at the rate of 6 feet a second. Find the direction and the magnitude of his actual velocity.

7. A ship is traveling at a speed of 20 knots. (A knot is a nautical mile per hour, a nautical mile being approximately 1.1516 statute miles of 5280 feet each.) When directly opposite a target it fires a gun whose projectile has a velocity of 2000 feet a second. At what angle with the direction of motion of the ship must the gun be pointed in order to hit the target?

8. An airplane which has a speed of 120 miles an hour in calm air is headed southeast. A wind having a velocity of 15 miles an hour is blowing from the southwest. (a) Find the magnitude and the direction of the velocity of the airplane with reference to the ground. (b) How must the airplane be pointed in order to fly southeast, and what will be its actual speed?

9. A weight of 150 pounds is placed on a smooth plane surface which makes an angle of 35° with the horizontal, as shown in Fig. 21. The weight is held in place by a string parallel to the surface and fastened at the top of the plane. Find the pull on the string.

SUGGESTION. The pull will be equal to the component of the 150-pound weight parallel to the plane.

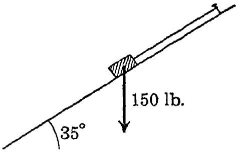


FIG. 21

10. A block is held in position on a smooth inclined plane by means of a string as in Fig. 21. If the pull on the string is 27.3 pounds, and the inclination of the plane is 24° 50′, what is the weight of the block?

omit

★10. Isosceles triangles and regular polygons.

Since the perpendicular from the vertex of an isosceles triangle divides it into two congruent right triangles, we can solve the isosceles triangle by solving one of these right triangles.

To solve a problem involving a regular polygon of n sides, we may first divide it into n congruent isosceles triangles.

Example 1.

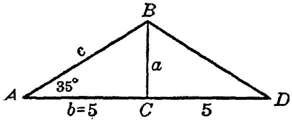


FIG. 22

A garage has a gable roof whose rafters make an angle of 35° with the horizontal. What is the length of a rafter if the width of the garage is 10 feet?

SOLUTION. In Fig. 22, AD represents the width of the garage and AB the length of the rafter.

$$\cos 35° = \frac{b}{c}, \qquad c = \frac{b}{\cos 35°} = \frac{5}{0.8192} = 6.1 \text{ ft.}$$

Example 2.

Find the length of the side of a regular pentagon inscribed in a circle of radius 6 inches.

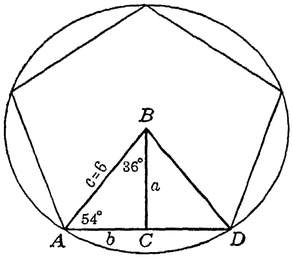


FIG. 23

SOLUTION. Each side of the pentagon subtends a central angle of $\frac{1}{5} \times 360° = 72°$. In Fig. 23, angle $ABC = \frac{1}{2} \times 72° = 36°$, and angle $BAC = 90° - 36° = 54°$. In triangle ABC,

$$\frac{b}{6} = \cos 54°, \qquad b = 6 \cos 54° = 6 \times 0.5878 = 3.527.$$

$$AD = 2b = 7.054 \text{ in.}$$

EXERCISES II. D

1. Each of the equal angles of an isosceles triangle is 40° 15′, the base is 15 inches. Find the remaining parts and the area.
2. Each of the equal sides of an isosceles triangle is 11.52 inches, the vertex angle is 32° 15′. Find the base.
3. The equal sides of a wedge are 4.2 inches, the base is 1.6 inches. Find the angles.

4. Find the radius of a circle in which a 59-foot chord subtends an angle of 12° at the center.
5. The radius of a circle is 40 inches, the length of a chord is 70 inches. Find the central angle subtended by the chord.
6. Find the radius of a circle in which a chord of 7.1 inches subtends an angle of 142° 36′ at the center.
7. Find the chord of a 35° arc in a circle of radius 14 inches.
8. Find the length of a belt passing around two pulleys whose radii are 14 inches and 5 inches respectively, and whose distance apart, between centers, is 10 feet.
9. A barn has a gable roof whose rafters are 20 feet long. The width of the barn is 30 feet. Find the angle that the rafters make with the horizontal. Find the area of one of the gable ends (i.e., the triangle in Fig. 24).

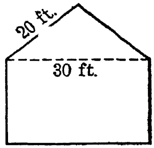


FIG. 24

10. A barn is 30 feet wide by 60 feet long; the rafters make an angle of 40° with the horizontal. Find the area of each of the two gable ends and the area of the roof.
11. Find the radius, the apothem (perpendicular distance from the center to a side), and the area of the following regular polygons: (a) a decagon whose side is 10 inches; (b) a 9-sided polygon whose side is 15 inches; (c) a 20-sided polygon whose side is 6.758 inches.
12. The radius of a circle is 100 feet. Find the perimeter and the area of (a) a regular inscribed pentagon; (b) a regular inscribed decagon; (c) a regular circumscribed pentagon; (d) a regular circumscribed decagon.
13. The area of a regular pentagon is 560 square feet. Find the radii of the circumscribed and inscribed circles.
14. A metal nut $\frac{3}{4}$ inch thick is in the shape of a regular hexagon, the distance between the parallel sides being $1\frac{3}{4}$ inches. The circular hole through the center is $\frac{3}{4}$ inch in diameter. Find the amount of metal in the nut.
15. Show that the area of a regular polygon of n sides circumscribed about a circle of radius r is

$$nr^2 \tan 180^\circ$$

16. Show that the perimeter of a regular polygon of n sides inscribed in a circle of radius r is

$$2nr \sin {}^{180^\circ}$$

★11. Solution of oblique triangles by means of right triangles.

Oblique triangles can always be solved by breaking them up into right triangles. The following examples illustrate the methods used in the four typical cases which arise. Usually, however, it will be found more convenient to employ other methods and formulas for solving oblique triangles. These will be developed in a later chapter.

Case I. Two angles and a side given.

Example 1.

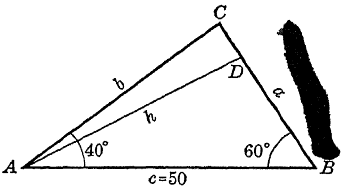


FIG. 25

In the triangle ABC, $A = 40°$, $B = 60°$, $c = 50$. Find the remaining parts.

SOLUTION. $C = 180° - (A + B) = 180° - (40° + 60°) = 80°$. Draw the altitude from one end of the known side. Suppose that this altitude is $AD = h$ (Fig. 25). Then, in the right triangle ABD, $h = 50 \sin 60° = 43.30$.

Now, in the right triangle ADC,

$$b = \frac{h}{\sin C} = \frac{43.30}{\sin 80°} = 44.0.$$

Side a may be found in a similar manner by drawing the altitude from B, or by computing the segments BD and DC and adding them.

Case II. Two sides and the angle opposite one of them given. (See discussion, section 53, pages 84–86.)

Example 2.

Given $A = 75°$, $a = 20$, $b = 10$; find B, C, c.

SOLUTION. Draw the altitude $CD = h$ (Fig. 26). (The altitude must not be drawn from the vertex of the known angle.) In the right triangle ADC,

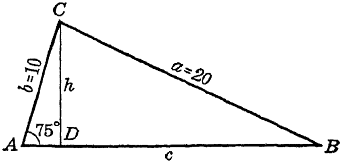


FIG. 26

$$h = b \sin A = 10 \sin 75° = 9.659.$$

In the right triangle BDC,

$$\sin B = \frac{h}{a} = \frac{9.659}{20} = 0.48295, \qquad B = 28° \, 53'.$$

$$C = 180° - (A + B) = 180° - 103° \, 53' = 76° \, 7'.$$

Side c may be similarly found by drawing the altitude from B, or by computing the segments AD and DB and adding.

Case III. Two sides and the included angle given.

Example 3.

Given $a = 25$, $b = 30$, $C = 50°$; find the other parts.

SOLUTION. Draw an altitude to one of the known sides, preferably the larger. Suppose that this altitude is $BD = h$, and that it divides the side BC into the segments $CD = m$ and $DA = n$ (Fig. 27). Then,

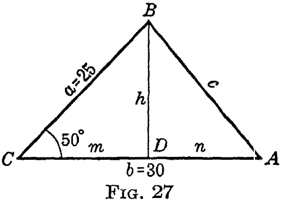


FIG. 27

$$h = a \sin C = 25 \sin 50° = 19.15,$$
$$m = a \cos C = 25 \cos 50° = 16.07,$$
$$n = b - m = 30 - 16.07 = 13.93,$$

$$c^2 = h^2 + n^2 = (19.15)^2 + (13.93)^2 = 560.8, \qquad c = 23.7.$$

Angles A and B can now be found quite easily.

Case IV. Three sides given.

Example 4.

The three sides of a triangle are $a = 5$, $b = 6$, $c = 9$. Find the angles.

SOLUTION. Draw an altitude to one of the sides, preferably the largest. Suppose that this altitude h divides the side AB into segments $AD = m$ and $DB = n$ (Fig. 28). Then,

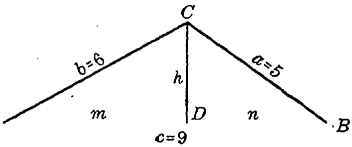


FIG. 28

$$h^2 = 36 - m^2 = 25 - n^2,$$
$$m^2 - n^2 = 36 - 25 = 11,$$
$$(m + n)(m - n) = 11.$$

But,

$$m + n = 9,$$

and consequently, $m - n = \dfrac{11}{9}$.

Solving these simultaneous equations, we get

$$m = \frac{46}{9}, \qquad n = \frac{35}{9}$$

$$\cos A = \frac{m}{b} = \frac{23}{27} = 0.8519, \qquad A = 31.6°;$$

$$\cos B = \frac{n}{a} = \frac{7}{9} = 0.7778, \qquad B = 39.0°;$$

$$180° - (A + B) = 180° - 70.6° = 109.4°.$$

EXERCISES II. E

Solve the following triangles:

✓ **1.** $A = 30°, B = 80°, a = 15.$ **2.** $A = 35°, b = 17, c = 32.$
3. $A = 70°, a = 8, c = 5.$ **4.** $B = 100°, C = 30°, b = 75.$
5. $a = 2.3, b = 1.5, c = 1.6.$ **6.** $a = 26, c = 40, B = 62°.$
7. $C = 100°, a = 82, c = 105.$ **8.** $a = 95, b = 102, c = 150.$

9. From the top of a hill, the angles of depression of two successive milestones on a level road, which leads straight away from the hill, are 5° and 15° respectively. Find the height of the hill.

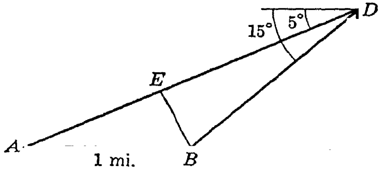


FIG. 29 (not drawn to scale).

SUGGESTION. In Fig. 29 BE is drawn perpendicular to AD. Find BE, then BD, finally CD.

10. At a certain horizontal distance from the foot of a vertical cliff, the angle of elevation of the top of a flagpole 50 feet tall standing on the edge of the cliff is 40°. From the same position, the angle of elevation of the foot of the pole is 35°. How high is the cliff?
11. At a certain point, the angle of elevation of the top of a flagpole, which stands on level ground, is 35°. Seventy-five feet nearer the pole, the angle of elevation is 50°. How high is the pole?
12. Solve the preceding exercise if the angles of elevation are 30° and 45° respectively.
13. From a window 30 feet above the street, the angle of depression of the curb on the near side of the street is 50°, that of the curb on the far side is 13°. How wide is the street from curb to curb?
14. At a point in the same horizontal plane with the foot of a vertical cliff 150 feet high, the angles of elevation of the top and the bottom of a flagpole standing on top of the cliff are 20° and 16° respectively. Find the height of the pole.
15. Points A and B are on opposite sides of a lake. At a point C, which is 456 feet from A and 580 feet from B, the angle subtended by the line AB is $44° 35'$. Find the distance from A to B.
16. The sides of a triangle are 20, 25, and 30. Find the length of the altitude to the longest side.

CHAPTER

Approximate Numbers and Computation

12. Approximate numbers.

An **approximate number** is one which differs slightly from the exact number for which it stands. In trigonometry we deal almost entirely with approximate numbers. With certain exceptions (e.g., $\sin 30° = \frac{1}{2} = 0.5$), all of the tabulated values of the trigonometric functions are approximations. Thus, when we write

$$\sin 45° = \frac{\sqrt{2}}{2} = 0.7071,$$

we do not mean that $\sin 45°$ is exactly equal to 0.7071, but that 0.7071 is the four-place number which is closest to the value of $\sin 45°$.

All measurements are approximate numbers. When we measure a line to the nearest tenth of an inch and say that its length is 18.3 inches, we mean that the length is between 18.25 inches and 18.35 inches.

13. Rounding off numbers.

It is often desirable to reduce an approximate number to one of less accuracy. This process is called **rounding off** the number. In rounding off a number we choose the nearest number having the desired number of places. Thus, if we round off 4.2537 to thousandths, we get 4.254. If we round it off to hundredths, we get 4.25.* To tenths, the number is 4.3.

* Here it would be best to write 4.25+. Similarly, in rounding off the

In rounding off a number ending in 5, to a number having one less digit, it is customary to *make the resulting number end in an even digit.* Thus, 17.25 becomes 17.2, while 17.75 becomes 17.8.

★14. Error.

The difference between an approximate value of a quantity and its exact or true value is the **absolute error** of the approximate value. In the approximate number 18.3, the maximum absolute error is 0.05, since 18.3 cannot be less than 18.25 or greater than 18.35. The **relative error** is the quotient of the absolute error divided by the true value. (Ordinarily the true value is not ascertainable, and we are forced to use the approximate value for the divisor. This does not make an appreciable difference in the quotient.) The maximum relative error in the example just given is $0.05/18.3 = 0.003$, or 0.3 per cent.

Relative error is independent of the position of the decimal point. Thus, a measurement of 1.83 inches, although accurate to hundredths, is relatively no more accurate than a measurement of 18.3 inches. For the maximum relative error of the approximate number 1.83 is $0.005/1.83 = 0.003$, and this is exactly the same as the maximum relative error of 18.3.

15. Significant figures.

The illustration of the preceding section indicates that relative accuracy does not depend upon the number of decimal places or upon the position of the decimal point, but upon the number of significant figures that the number contains. A **significant figure** is any one of the digits from 1 to 9 inclusive, and 0 except when it is used to fix the decimal point or to fill the places of unknown or discarded digits.

number 6.347, it would be best to write 6.35 –. This is helpful if the number is to be rounded off still further.

The 0's in 0.75 and 0.0024 are not significant figures.

The 0 in 6.80 is a significant figure. In this connection, note that 6.80 means a number between 6.795 and 6.805, whereas 6.8 means a number between 6.75 and 6.85. The number 6.80 has three significant figures, and is more accurate than 6.8, which has only two.

The significance of 0's at the right of a whole number is doubtful. For example, if it is stated that a man's income for a certain calendar year is $5000, it is impossible to say, without further information, which, if any, of the 0's are significant figures. If his income tax return were available and showed his income to be $5043.75, the first 0 in the $5000 would be significant but the other two would not. If the return showed his income to be $5122.80, none of the 0's in the $5000 would be significant.

16. Scientific notation.

The **leading digit** of a number is the first non-zero digit from the left (i.e., the first significant figure). A number is said to be expressed in **scientific notation** when it is written as the product of a number having the decimal point just after the leading digit, and a power of 10. (When the decimal point is just after the leading digit it may be said to be in **standard position.**)

The method of changing from the usual to the scientific notation is illustrated by the following examples:

$$237.65 = 2.3765 \times 100 = 2.3765 \times 10^2,$$
$$0.0054 = 5.4 \div 1000 = 5.4 \times 10^{-3}.$$

It is possible to indicate, by writing a number in scientific notation, whether the 0's at the right of a number are significant. Thus, if in the number 1,300,000 the first two 0's are significant but the last three are not, we could write the number in the form 1.300×10^6.

EXERCISES III. A

1. Round off the following numbers to one less decimal place: 12.34, 29.87, 4.06, 1.396, 0.251, 0.215, 68.2, 63.25, 1.9999, 1.9995, 2.355, 2.345, 2.354, 2.350.
2. Round off the following numbers (a) to three decimal places, (b) to three significant figures: 1.2464, 0.5864, 12.9065, 12.9055, 2.3505, 16.0031, 0.003664.
3. Find the maximum relative error in each of the following approximate numbers: 24.2, 105.16, 38.985, 0.002, 0.00025.
4. How many significant figures are there in each of the following numbers? 39.46, 1.004, 1.400, 0.0014, 100.03, 0.00005, 123892, 200.0.
5. Underline the significant 0's in the following numbers, and put a question mark under each doubtful 0: 10.02, 10.20, 0.20, 0.02, 0.020, 25000, 2506, 0.00300, 0.20500, 20500.
6. Express the following numbers in scientific notation: 256835, 0.000232, 0.000,000,006, 3876.5, 984.876, 1,462,817.
7. Write each of the following numbers in ordinary notation: 1.8×10^7, 2.35×10^{-5}, 8.482×10^8, 3.7×10^{-9}.

*17. Addition and subtraction of approximate numbers.

When two or more approximate numbers are added, the sum cannot be more accurate than the least accurate of the numbers. (This is in the sense of absolute accuracy, not relative accuracy.) For example, consider the sum of the numbers 2.3683, 81.02, 0.0457. The sum cannot be accurate beyond hundredths, so some of the numbers can be rounded off. We carry them, whenever possible, to one more place than the least accurate number, on the theory that the errors in these numbers tend to compensate for each other (that is, that positive and negative errors occur in nearly equal proportions). Thus, we write

$$\begin{array}{r} 2.368 \\ 81.02 \\ 0.046 \\ \hline 83.434 \end{array}$$

The sum should be rounded off to hundredths, giving 83.43.

The above remarks apply also to subtraction.

*18. Multiplication of approximate numbers.

Suppose that the sides of a rectangle are measured as 5.73 and 6.42 inches respectively. The area would be found by multiplying these numbers together; thus,

$$\text{area} = 5.73 \times 6.42 = 36.7866.$$

However, this result is not accurate to as many significant figures as are given. For the approximate number 5.73 means some value between 5.725 and 5.735; similarly, 6.42 means a value between 6.415 and 6.425. Therefore we can merely say that the area is between

$$\begin{aligned} & 5.725 \times 6.415 = 36.725875, \\ \text{and} \quad & 5.735 \times 6.425 = 36.847375. \end{aligned}$$

Therefore, in the product 36.7866 we retain only three significant figures, namely 36.8; even then the last digit is not absolutely certain.

In general, we are not justified in retaining more significant figures in a product calculated from approximate numbers than the least accurate of the factors which go to make up the product. Thus, we round off all the factors to the number of such figures in the least accurate factor. The multiplication can then be performed in contracted form, in which the partial products are carried just one place beyond the last place which is to be retained. The following illustration of the multiplication of 6.42 by 5.73 exhibits the method:

$$\begin{array}{r} 6.42 \\ 5.73 \\ 32.10 \\ 4.49 \\ .19 \\ \hline 36.78 \end{array}$$

The first partial product is obtained by multiplying the multiplicand, 6.42, by the leading digit, 5, of the multiplier; thus, $5 \times 6.42 = 32.10$.

Multiplying by the next digit of the multiplier, we have $7 \times 2 = 14$, and we should write the 4 one place to the right of the 0 in 32.10, and on the next line below, carrying the 1. However, we do not write down the 4, as it does not contribute to the accuracy of our final product, but merely carry the 1. In this way, we find 4.49 as our second partial product.

Before finding our third partial product, we strike out the 2 in the multiplicand. Then we find that $3 \times 4 = 12$, and carry the 1 to add to 3×6. Thus, the third partial product is .19.

The sum of the partial products is rounded off to three significant figures, giving 36.8 as the final product.

*19. Division of approximate numbers.

As in multiplication, so in division, we can show that in general it is useless to retain more figures in the quotient than the number of significant figures in the less accurate of the two numbers, dividend and divisor. Consequently, we note which of these contains the fewer significant figures, and round the other off to the same number of such figures. If, after this has been done, the dividend, *without regard to the decimal point*, is less than the divisor, we restore one digit to the dividend. (See example below.) The quotient is carried to the same number of significant figures as are contained in the divisor. A contracted form of the division process as applied to the example $36.78 \div 6.42$ is shown on page 42.

Here, if the dividend were rounded off to 368 (decimal point omitted), it would be less than the divisor, 642. Hence, we retain four, rather than three, figures in the dividend.

$$
\begin{array}{r@{}l}
 & 5.73 \\
6.42) & \overline{36.78} \\
 & \underline{32\ 10} \\
 & 4\ 68 \\
 & \underline{4\ 49} \\
 & 19 \\
 & \underline{19}
\end{array}
$$

After the first partial product ($5 \times 642 = 3210$) has been subtracted, we do not bring down a 0 from the dividend, but strike out the final digit, 2, in the divisor.

The next digit in the quotient will obviously be 7. We note that $7 \times 2 = 14$, but do not write down the 4; we merely carry the 1. The partial product is 449.

The process is continued as far as possible, cutting down the divisor by one digit at each stage. The final quotient is 5.73.

★20. Square root.

It will be assumed that the student is familiar with the method of extracting square root learned in arithmetic. How a table of squares, such as is to be found in Table VI of the Macmillan Logarithmic and Trigonometric Tables, can be used to expedite the process will be illustrated by extracting the square root of 1350 (considered as an exact, not an approximate, number).

$$
\begin{array}{r@{}l}
 & 1\acute{3}5\acute{0}.0\acute{0}\ (36.7 \\
(367)^2 = & 1346\ 89 \\
2 \times 367 = 734 & \overline{|\ 3\ 11}
\end{array}
$$

After separating the number into groups of two digits each, starting at the decimal point and going both to left and to right, we note that the largest square contained in the group at the left, namely 13, is the square of 3. Turning to the 300's of Table VI, we find that the largest square just below 135000 is 134689, which is the square of 367.

Subtracting the square of 367, we have a remainder of 311. This is the process previously learned, except that we have subtracted the square of a three-digit number instead of that of a one-digit number.

The process may now be continued as usual. It may be noted, however, that if we have obtained k significant figures in the square root, then $k - 1$ more may be obtained by division. Thus, in the present example, we may divide 311 by 734 and obtain two more significant figures in the square root.

★21. Use of calculating machines.

If a calculating machine is available, the contracted forms of multiplication and division are of course not used. All that has been said about significant digits, however, holds. For example, it would be absurd to carry the quotient of $36.78 \div 6.42$ out to eight or ten figures, even though the division could easily be performed on a machine.

While it is possible to extract square root on a calculating machine, an effective method is to use a table of squares, such as Table VI,* in conjunction with a machine, employing the machine to perform the final division.

EXERCISES III. B

Perform the following operations, retaining the proper number of significant figures:

1. 35.8×41.6.
2. 5.25×48.4.
3. 14.26×3.860.
4. 529.6×29.64.
5. 5028×46.09.
6. 0.1283×127400.
7. $43.8 \times 13.1 \times 32.8$.
8. $0.532 \times 0.00567 \times 12.3$.
9. 13845×89.763.
10. $7.283 \times 283.4 \times 5.437$.
11. $63.1 \div 21.5$.
12. $0.5929 \div 3.801$.
13. $52.96 \div 1.895$.
14. $2.451 \div 1903$.
15. $2500 \div 16.98$.
16. $32.17 \div 712.3$.
17. $(436.5)^2$.
18. $(71.48)^2$.

* Or Barlow's Tables.

19. $\dfrac{35.8 \times 9.86}{136}$ **20.** $\dfrac{12.34 \times 1.986}{286.4}$

Extract the square roots of the following quantities, carrying the results to four significant figures:

21. 1683. **22.** 25648. **23.** 17.986.
24. 0.01534. **25.** 0.6843. **26.** 1.0076.

CHAPTER IV

Logarithms

22. Logarithms.

The ***logarithm*** *of a number to a given base is the exponent of the power to which the base must be raised to yield the number.* It is assumed that the base is positive and different from 1, and that the number is positive.

Thus, since $2^3 = 8$, 3 is the logarithm of 8 to the base 2. This may be written in the form $\log_2 8 = 3$. More generally, we write

$$\log_b N = x, \tag{1}$$

where $$b^x = N \qquad (b > 0, \neq 1; N > 0). \tag{2}$$

Forms (1) and (2) are equivalent.

The base in most common use is 10. Since, for example, $10^2 = 100$, we have $\log_{10} 100 = 2$. As we shall deal almost exclusively with logarithms to the base 10 (that is, **common logarithms**), we shall omit the subscript indicating the base, and write simply $\log 100 = 2$. Thus,

$$\begin{array}{llcll}
10^3 = 1000, & \text{or} & \log 1000 & = & 3; \\
10^2 = 100, & \text{or} & \log 100 & = & 2; \\
10^1 = 10, & \text{or} & \log 10 & = & 1; \\
10^0 = 1, & \text{or} & \log 1 & = & 0; \\
10^{-1} = 0.1, & \text{or} & \log 0.1 & = & -1; \\
10^{-2} = 0.01, & \text{or} & \log 0.01 & = & -2; \\
10^{-3} = 0.001, & \text{or} & \log 0.001 & = & -3.
\end{array}$$

The logarithms of integral powers of 10, such as the foregoing, can, because of the very meaning of logarithm, be

expressed exactly. Although the logarithm of a number such as 3, for example, cannot be expressed exactly in the decimal notation, we assume that a number x exists for which $10^x = 3$, and that an approximation to this number can be found. Actually, such an approximation, to five decimal places, is 0.47712, and we write $\log 3 = 0.47712$. Similarly, $\log 3.262 = 0.51348$. (How these values are obtained from tables will be explained later.)

23. Mantissa.

Assuming that

$$\log 3.262 = 0.51348,$$

let us write

$$10^{0.51348} = 3.262. \tag{1}$$

Multiplying both sides by 10, we get

$$10^{1.51348} = 32.62,$$

which, in logarithmic notation, is

$$\log 32.62 = 1.51348.$$

By dividing both sides of (1) by 10, we get

$$10^{0.51348-1} = 0.3262,$$

or

$$\log 0.3262 = 0.51348 - 1.$$

This could also be written $\log 0.3262 = -0.48652$,* but it is usually more convenient to keep the decimal part of a logarithm positive. This positive decimal part of a logarithm is called the **mantissa** of the logarithm.

The two examples given above illustrate the fundamental principle: *For numbers having the same sequence of digits,* such as 3.262, 32620, 0.003262, *the mantissa of the logarithm is the same.*†

* Found by subtracting 0.51348 from 1 and prefixing a negative sign.
† Provided that the base is 10.

24. Characteristic.

The integral, or whole-number, part of a logarithm is called the **characteristic.** Thus, since log 32.62 = 1.51348, the characteristic of the logarithm of 32.62 is 1.

Since log 1 = 0, and log 10 = 1, the logarithm of a number between 1 and 10, for example 3.262, is between 0 and 1 in value, and consequently has the characteristic 0.* We shall say that such a number has the decimal point in **standard position,** namely after the first non-zero digit. (See section 16.)

Each time we multiply a number by 10 we move the decimal point one place to the right, and each time we divide by 10 we move the point one place to the left. But each time we multiply a number by 10 we increase the logarithm of the number by 1, and each time we divide a number by 10 we decrease its logarithm by 1, as was seen in the illustration above. Thus, we may state the following rule for finding the characteristic:

If a number has its decimal point in standard position (i.e., after the first non-zero digit), *the characteristic of the logarithm of the number is zero; if the decimal point is not in standard position, the characteristic of the logarithm of the number is equal to the number of places the point has been moved from standard position, and is positive if the point has been moved to the right, negative if it has been moved to the left.*†

For example, in the number 78460, the decimal point has been moved from standard position (after the 7) 4 places to the right (after the 0), and the characteristic of the logarithm of the number is therefore 4.

In the number 0.03262, the point has been moved from standard position 2 places to the left. The characteristic of the logarithm of the number is therefore −2. In fact,

* A characteristic should always be written, even though it is 0.

† Note that *the characteristic is also equal to the exponent of 10 when the number is written in scientific notation.* (See section 16.)

since we saw above that $\log 3.262 = 0.51348$, we may write

$$\log 0.03262 = 0.51348 - 2.$$

It is frequently convenient to write this in the form

$$\log 0.03262 = 8.51348 - 10.$$

The rule given for determining the characteristic also tells us how to point off a number corresponding to a given logarithm. (The number corresponding to a logarithm is called the **antilogarithm.** More precisely, if $\log N = x$, then N is the antilogarithm of x.)

Thus, if we have given

$$\log N = 2.51348,$$

we know from the illustration above that the number N is composed of the sequence of digits 3262. Since the characteristic is 2, the decimal point has been moved 2 places to the right from standard position. Therefore,

$$N = 326.2.$$

EXERCISES IV. A

Determine the characteristic of the logarithm of:

1. 436.	**2.** 25.	**3.** 3280.
4. 4.	**5.** 0.136.	**6.** 0.2.
7. 0.42.	**8.** 0.04.	**9.** 0.0075.
10. 1.0075.	**11.** 0.1075.	**12.** 52.684.
13. 21.64.	**14.** 384.6.	**15.** 2500.
16. 0.384.	**17.** 8.124.	**18.** 0.2960.
19. 380000.	**20.** 0.006934.	**21.** 0.02796.
22. 7.952.	**23.** 98.	**24.** 98.5.
25. 98.52.	**26.** 985.	**27.** 9852.
28. 0.9852.	**29.** 0.985.	**30.** 0.98.
31. 0.098.	**32.** 0.000,001,2.	**33.** 60,000,000.
34. 6.	**35.** 0.6.	**36.** 0.600.

25. Finding the mantissa.

In a standard five-place table of logarithms, such as Table I of the Macmillan Logarithmic and Trigonometric Tables, the first three digits of a number are found at the left of the page, the fourth digit at the top or bottom, the corresponding mantissa (decimal point omitted) being in the same row as the first three digits of the number and in the same column as the fourth digit. The student should verify that the mantissa of the logarithm of 3262 is .51348.

To find the logarithm of a number composed of five digits we must use interpolation. (See section 8.)

Example.

Find log 32.627.

Solution. Find the mantissas for the numbers next above and next below 32.62:

		Number	**Mantissa** (decimal point omitted)		
.010		32.630	51362		14
	.007	32.627	?	x	
		32.620	51348		

Assuming that the change in the mantissa is proportional to the change in the number,* we have

$$\frac{x}{14} = \frac{0.007}{0.010} = 0.7,$$

$$x = 0.7 \times 14 = 9.8.$$

$$\text{Mantissa} = 51348 + 10 = 51358.$$

$$\log 32.627 = 1.51358.$$

Once the principle of proportionality or proportional parts is understood, the work can be arranged more com-

* This is only approximately true.

pactly in some such way as the following, or may be performed mentally.

$$\begin{array}{rr} 32.63 \sim & 51362 \\ 32.62 \sim & \underline{51348} \\ \text{difference} = & 14 \\ & \times \underline{0.7} \\ & 9.8 \\ & 51348 \\ \log 32.627 = & 1.51358 \end{array}$$

(The symbol $\sim$ may here be read "corresponds to.")

EXERCISES IV. B

Find the logarithm of each of the following numbers:

1. 68.	**2.** 68.3.	**3.** 359.
4. 381.	**5.** 2.	**6.** 2.87.
7. 5000.	**8.** 751.5.	**9.** 8428.
10. 0.4313.	**11.** 0.02156.	**12.** 56980.
13. 250000.	**14.** 0.00036.	**15.** 7.851.
16. 1.003.	**17.** 15.95.	**18.** 0.003097.
19. 2.9645.	**20.** 23572.	**21.** 6784.8.
22. 67.843.	**23.** 54326.	**24.** 38.794.
25. 6.3129.	**26.** 0.34732.	**27.** 0.000,876,95.
28. 1.0006.	**29.** 9.9982.	**30.** 99.992.
31. 99998.	**32.** 0.10101.	**33.** 0.000,100,01.
34. 2509.9.	**35.** 829.99.	**36.** 91.119.

26. Finding the antilogarithm.

The process of finding the number corresponding to a given logarithm is illustrated by the following examples:

Example 1.

Find the number whose logarithm is $7.91121 - 10$.

SOLUTION. The mantissa is found exactly in the table. At the left we find 815; at the top we find 1. Thus, the number is composed of the sequence of digits 8151. The characteristic is $7 - 10 = -3$. Consequently, the decimal point must be moved from

standard position (after the 8) 3 places to the left. Therefore the number is 0.008151.

Example 2.

Given $\log N = 1.91123$; find N.

SOLUTION. Here we use inverse interpolation.

Mantissa	Number
91126	8152
91123	?
91121	8151

$$5\left[\begin{array}{l} 91126 \\ 2\left[\begin{array}{l} 91123 \\ 91121 \end{array}\right. \end{array}\right. \qquad \left.\begin{array}{r} 8152 \\ \left.\begin{array}{r} ? \\ 8151 \end{array}\right]x \end{array}\right]1$$

$$\frac{x}{1} = \frac{2}{5} = 0.4.$$

$$N = 81.514.$$

EXERCISES IV. C

Find the number corresponding to each of the following logarithms:

1. 0.69897.
2. 1.76042.
3. 2.93601.
4. 4.26174.
5. 0.81278 − 1.
6. 9.96741 − 10
7. 3.76253 − 10.
8. 3.63337.
9. 8.84442 − 10
10. 0.63994.
11. 0.69085 − 2.
12. 1.51416.
13. 7.19767 − 10.
14. 1.48762.
15. 8.82326 − 10
16. 5.18752.
17. 6.15465.
18. 9.79029 − 10
19. 0.83445.
20. 6.36021 − 10.
21. 1.94548.
22. 9.00000 − 10.
23. 1.00009.
24. 0.99998.

27. Laws of logarithms.

Since logarithms are exponents, they obey the laws of exponents, it being assumed that these laws hold for irrational as well as rational exponents.*

I. The logarithm of a product is equal to the sum of the logarithms of its factors.

* See the author's *College Algebra.*

Let $\log_b M = x, \qquad \log_b N = y.$

Then, $M = b^x, \qquad N = b^y,$

$$MN = b^x b^y = b^{x+y},$$

$$\log_b MN = x + y,$$

or $$\boldsymbol{\log_b MN = \log_b M + \log_b N.}$$

The proof can easily be extended to cover the case of any finite number of factors.

II. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

Using the same notation as above, we have

$$\frac{M}{N} = \frac{b^x}{b^y} = b^{x-y},$$

$$\log_b \frac{M}{N} = x - y,$$

or $$\boldsymbol{\log_b \frac{M}{N} = \log_b M - \log_b N.}$$

III. The logarithm of the mth power of a number is equal to m times the logarithm of the number.

If $\log_b N = x$, then $N = b^x$, and

$$N^m = (b^x)^m = b^{mx},$$

$$\log_b N^m = mx,$$

or $$\boldsymbol{\log_b N^m = m \log_b N.}$$

IV. The logarithm of the mth real positive root of a number is equal to one mth of the logarithm of the number.

This is really the same as III, since $\sqrt[m]{N} = N^{1/m}$. Thus,

$$\boldsymbol{\log_b \sqrt[m]{N} = \frac{1}{m} \log_b N.}$$

28. Logarithmic computation of products and quotients.

The advantage of logarithms in performing multiplication and division is that these operations can be replaced

by the simpler operations of addition and subtraction respectively.

It must be realized that results are only approximate.

Example 1.

Find the value of 32.62×8.673.

SOLUTION. Denoting the product by x, we have

$$\log x = \log 32.62 + \log 8.673.$$

We arrange the work as follows:

$$\begin{array}{rl} \log 32.62 & 1.51348 \\ \log 8.673 & 0.93817 \\ \hline \text{sum} = \log x & 2.45165 \\ x & 282.9 \end{array}$$

Example 2.

Find the value of $8.673 \div 32.62$.

SOLUTION. Let the quotient be denoted by x. Then

$$\log x = \log 8.673 - \log 32.62.$$

$$\begin{array}{rl} \log 8.673 & 0.93817 \\ \log 32.62 & 1.51348 \\ \hline \end{array}$$

Here we are subtracting the larger quantity from the smaller. In order to keep the mantissa positive, we add 10 to, and subtract 10 from, the logarithm of 8.673, getting

$$\begin{array}{r|l} \log 8.673 & 10.93817 - 10 \\ \log 32.62 & 1.51348 \\ \hline \text{difference} = \log x & 9.42469 - 10 \\ x & 0.2659 \end{array}$$

Example 3.

Find the value of

$$\frac{3262 \times 1.786}{532.1 \times 0.8673}.$$

SOLUTION. We note that

$$\text{log fraction} = \text{log numerator} - \text{log denominator},$$

and arrange the work as follows:

log 3262	3.51348	log 532.1	2.72599
log 1.786	0.25188	log 0.8673	9.93817 − 10
log numerator	3.76536	log denominator	12.66416 − 10
log denominator	2.66416		
log fraction	1.10120		
fraction	12.62		

Note that we do not interpolate to find a fifth figure in the antilogarithm because of the rules for computation with approximate numbers.

29. Cologarithm.

When one number is to be divided by another we may change the problem to one of multiplication by using the reciprocal of the divisor. For example, $3 \div 2 = 3 \times \frac{1}{2}$.

The logarithm of the reciprocal of a number is called the **cologarithm** of the number and is abbreviated **colog.** That is,

$$\text{colog } N = \log \frac{1}{N} = \log 1 - \log N = -\log N.$$

Thus, *the cologarithm of a number is the negative of the logarithm of the number.* Consequently, in solving a problem in division by means of logarithms, we may either subtract the logarithm of the divisor or add its cologarithm. There is no advantage, but rather a disadvantage, in using the cologarithm when only two numbers are involved in a division problem. There is, however, some advantage, particularly in the arrangement of the solution, when more than one number occurs in the denominator of an expression.

The cologarithm of a number is obtained by subtracting

the logarithm of the number from log 1, that is, from 0. The 0 is usually written in the form 10 − 10, and the subtraction can be performed mentally, after some practice, by the following method: *Begin at the left, and subtract from 9 each digit of the logarithm except the last non-zero digit, which must be subtracted from 10.*

Examples.

log 32.62 = 1.51348,	log 0.01508 = 8.17840 − 10,
colog 32.62 = 8.48652 − 10,	colog 0.01508 = 1.82160.

Following is a solution of example 3 above which employs cologarithms:

log 3262	3.51348
log 1.786	0.25188
colog 532.1	7.27401 − 10
colog 0.8673	0.06183
log fraction	11.10120 − 10
fraction	12.62

30. Logarithmic computation of powers and roots.

The operations of raising to powers and of extracting roots are greatly facilitated by the use of logarithms, because it replaces these operations by the simpler ones of multiplication and division.

Example 1.

Evaluate $(3.262)^4$.

Solution. Let $x = (3.262)^4$; then $\log x = 4 \log 3.262$.

log 3.262	0.51348
	× 4
log x	2.0539*
x	113.2

* Only five significant figures are retained here because of the rules for computation with approximate numbers.

Example 2.

Find the cube root of 3.262.

SOLUTION. If x is the desired cube root, then

$$\log x = \frac{1}{3}\log 3.262.$$

log 3.262	0.51348 (÷3
log x	0.17116
	1.4831

Example 3.

Find the cube root of 0.3262.

SOLUTION. If x is the desired cube root, then

$$\log x = \frac{1}{3}\log 0.3262 = \frac{1}{3}(9.51348 - 10).$$

In order to make the negative part of the characteristic exactly divisible by 3, add 20 and subtract 20:

log 0.3262	29.51348 − 30 (÷3
log x	9.83783 − 10
x	0.68838

EXERCISES IV. D

Find the value of each of the following expressions by means of logarithms:

1. 41.6×35.8.
2. 4.84×5.25.
3. $41.6 \div 35.8$.
4. $4.84 \div 5.25$.
5. 529.6×29.64.
6. 127400×0.1283.
7. 123.4×9.866.
8. $(3.482)^3$.
9. $5.832 \div 25.96$.
10. $7.283 \times 283.4 \times 5.437$.
11. $\sqrt{26.18}$.
12. $\sqrt[3]{1.546}$.
13. $\sqrt{0.9146}$.
14. $\sqrt[5]{3}$.
15. 24284×3789.5.
16. $0.82371 \times 0.001{,}985{,}7$.
17. $1.3336 \div 2.1248$.
18. $1.7321 \div 0.73205$.
19. $0.41831 \div 0.057864$.
20. $48.252 \times 9.6384 \times 0.96384$.

21. $53201 \times 56784 \times 12619$.

22. $472.48 \times 45.990 \times 0.87723$.

23. $\sqrt{89897}$.

24. $\sqrt[3]{4.6123}$.

25. $\sqrt[3]{0.92468}$.

26. $\sqrt[3]{0.092468}$.

27. $\dfrac{9.812 \times 18.76}{405.1}$.

28. $\dfrac{32.64}{19.23 \times 0.7191}$.

29. $\dfrac{54.321 \times 2.7183}{3.1416}$.

30. $\dfrac{1776.4}{24.683 \times 1.0054}$.

31. $(648.35)^5$.

32. $(648.35)^{1/5}$.

33. $\sqrt{5.2683 \times 0.84216}$.

34. $(1.0025)^{-1/2}$.

35. $\dfrac{538.21 \times 1.7864}{0.40752 \times 863.76}$.

36. 97.304×71.486 [illegible]

37. $\dfrac{\sqrt[3]{25.321}}{\sqrt{1.0048}}$.

38. ([illegible] $)^{3/4}$

39. $0.15630(-3.6251)^3$ [illegible]

Note. Although negative numbers have no real logarithms, we can treat this problem as if all the numbers involved were positive, and then prefix the proper sign to the result. Here we have, symbolically,

$$\frac{(+)(-)^3}{(-)\sqrt[5]{-}} = \frac{(+)(-)}{(-)(-)} = \frac{-}{+} = -$$

Thus, a negative sign should precede the final result.

40. $(-1.2381)^2 \div (-7.9564)^3$.

41. $\sqrt[3]{-9999.4}$.

42. $\dfrac{6.8213 \times (-3.4868)}{12.863}$

43. $\dfrac{(-25.868)^2\sqrt[3]{-0.88255}}{-32.759}$.

31. Logarithms of the trigonometric functions.

Tables giving the values of the trigonometric functions of angles are called tables of "natural functions" to distinguish them from tables which give the logarithms of these functions. We might in all cases find the natural function, and then the logarithm of that function from a table of logarithms of numbers. However, we have tables

which omit one step in this process by giving the logarithm of the function directly, when the value of the angle is known (e.g., Table III of the Macmillan Logarithmic and Trigonometric Tables).

The process of finding the value of the logarithm of a trigonometric function is quite like that of finding the value of the natural function, even when interpolation is required. Similarly, the process of finding the angle, when the logarithm of the function is given, is in no respect different from that of finding the angle when the natural function is given.

Example 1.

Find log cos 17° 25.8′.

SOLUTION. The interpolation can be carried out as in section 8, or it can be arranged as follows (cf. section 25):

$$\begin{aligned} \log\cos 17^\circ\, 25' &= 9.97962 - 10 \\ \log\cos 17^\circ\, 26' &= \underline{9.97658} - 10 \\ \text{difference} &= 4 \\ &\ \ \underline{\times\ 0.8} \\ &3.2 \end{aligned}$$

$$\begin{aligned} \log\cos 17^\circ\, 25' &= 9.97962 - 10 \\ \text{negative correction} &= \underline{3} \\ \log\cos 17^\circ\, 25.8' &= 9.97959 - 10 \end{aligned}$$

Example 2.

Given log tan A = 0.10860; find the acute angle A.

SOLUTION.

$$1' \left\{ \begin{array}{l} \log\tan 52^\circ\, 6' = 0.10875 \\ x \left\{ \begin{array}{l} \log\tan A = 0.10860 \\ \log\tan 52^\circ\, 5' = 0.10849 \end{array} \right\} 11 \end{array} \right\} 26$$

$$\frac{x}{1'} = \frac{11}{26}, \qquad x = \frac{11}{26} \times 1' = 0.4'.$$

$$A = 52^\circ\, 5.4'.$$

EXERCISES IV. E

Find the following by using tables of logarithms of the trigonometric functions:

1. log sin 29°.
2. log cos 31°.
3. log sin 78° 10′.
4. log tan 74° 20′.
5. log cot 17° 17′.
6. log cot 80° 22′.
7. log tan 12° 25′.
8. log sin 31° 52′.
9. log cos 49° 12′.
10. log sin 6° 31′.
11. log sin 7° 46′.
12. log cos 53° 21′.
13. log cot 30° 26′.
14. log sin 26° 45′.
15. log tan 35° 15.3′.
16. log sin 12° 13.2′.
17. log cos 58° 37.8′.
18. log cot 81° 25.1′.
19. log sin 9° 41.4′.
20. log tan 54° 22.2′.
21. log sin 57° 17.7′.
22. log cos 45° 2.3′.
23. log cot 10° 59.9′.
24. log tan 88° 59.8′.

Find the acute angle A, given that

25. $\log \sin A = 9.53888 - 10$.
26. $\log \cos A = 9.99484 - 10$.
27. $\log \tan A = 0.30575$.
28. $\log \cot A = 1.54493$.
29. $\log \tan A = 0.18762$.
30. $\log \sin A = 9.71708 - 10$.
31. $\log \tan A = 9.28875 - 10$.
32. $\log \cos A = 9.53871 - 10$.
33. $\log \cos A = 9.72868 - 10$.
34. $\log \cos A = 9.88150 - 10$.
35. $\log \cos A = 9.89530 - 10$.
36. $\log \sin A = 8.90150 - 10$.
37. $\log \sin A = 9.80070 - 10$.
38. $\log \sin A = 9.99483 - 10$.
39. $\log \cot A = 9.18854 - 10$.
40. $\log \cot A = 0.18750$.
41. $\log \tan A = 0.06735$.
42. $\log \tan A = 0.10235$.
43. $\log \tan A = 1.55553$.
44. $\log \cot A = 8.99983 - 10$
45. $\log \sin A = 9.99950 - 10$.
46. $\log \tan A = 1.00000$.
47. $\log \cos A = 0.17182$.
48. $\log \sin A = 0.11111$.

Find, by using logarithms, the value of each of the following expressions:

49. 12.38 sin 13° 20′.
50. 485.6 cos 22° 28′.
51. 204.65 sin 28° 18.2′.
52. 98.128 tan 33° 35.6′.
53. 0.18622 cos 14° 8.3′.
54. 57663 cot 40° 40.8′.
55. 152.98 sin 74′ 22.9′.
56. 3004.2 tan 66° 33.4′.
57. 1.2346 cos 45° 45.4′.
58. 19.897 sin 38° 59.6′.

59. $\dfrac{543.21 \sin 72^\circ\ 14.3'}{\sin 22^\circ\ 18.9'}$.

60. $\dfrac{2381.4 \tan 44^\circ\ 18.3'}{4561.8}$

Find the value of the acute angle A, given that

61. $\sin A = \dfrac{548.26 \sin 75^\circ\ 43.3'}{865.27}$.

62. $\sin A = \dfrac{9753.6 \sin 18^\circ\ 36.6'}{8910.4}$.

CHAPTER V

Logarithmic Solution of Right Triangles

32. Logarithmic solution of right triangles.

The general instructions of section 7 apply to the logarithmic solution of right triangles. It should be noted that the theorem of Pythagoras is not adapted to the use of logarithms if it is written in the form $c^2 = a^2 + b^2$. However, if the hypotenuse, c, is one of the known parts, we can write

$$a^2 = c^2 - b^2 = (c + b)(c - b), \quad \text{or} \quad b^2 = (c + a)(c - a),$$

and to these forms logarithms can be applied.

An outline, like that in the model solution shown on page 62, should first be made out. Begin with the known parts and conclude with the check. The outline should be complete before any numerical values are written in.

The following general rules will be of use in determining the degree of accuracy to be expected *when dealing with approximate numbers*, not only in connection with right triangles, but for all trigonometric work:

Lengths expressed to two significant figures call for angles to be expressed to the nearest 30′, and vice versa.

Lengths expressed to three significant figures call for angles to be expressed to the nearest 5′, and vice versa.

Lengths expressed to four significant figures call for angles to be expressed to the nearest minute, and vice versa.

Lengths expressed to five significant figures call for angles to be expressed to the nearest tenth of a minute, and vice versa.

It is thus convenient, in dealing with lengths expressed

to three significant figures and angles expressed to the nearest 5′, to use a four-place table of natural functions, such as the table on pages 12–14, without interpolation, or with very rough interpolation. For lengths expressed to four significant figures and angles to the nearest minute, four-place tables of the natural functions or four-place logarithmic tables may be used; in either case interpolation should be employed. Also, for this degree of accuracy, five-place logarithmic tables may be used without interpolation. For lengths expressed to five significant figures and angles to the nearest tenth of a minute, five-place logarithmic tables should be used with interpolation.

Example.

Solve the right triangle in which $a = 16.84$, $c = 20.36$.

SOLUTION.

$\sin A = \dfrac{a}{c}$,	a	16.84	
	c	20.36	
$\log \sin A = \log a - \log c$.	$\log a$	1.22634	
	$\log c$	1.30878	
	$\log \sin A$	9.91756	10
	A	**55° 48′**	
$B = 90° - A$.	B	**34° 12′**	
	$c + a$	37.20	
$b = \sqrt{(c + a)(c - a)}$,	$c - a$	3.52	
	$\log(c + a)$	1.57054	
$\log b = \frac{1}{2}[\log(c + a) + \log(c - a)]$.	$\log(c - a)$	0.54654	
	$\log b^2$	2.11708	
	$\log b$	1.05854	
CHECK.	b	**11.44**	
$b = c \cos A$,	$\log c$	1.30878	
$\log b = \log c + \log \cos A$.	$\log \cos A$	9.74980	
	$\log b$	1.05858	

The work is checked, since the values of $\log b$, obtained by two different methods, agree except in the last place.

EXERCISES V. A

Find the remaining parts, and also the areas, of the following right triangles ($C = 90°$) by logarithms:

1. $a = 793.6$, $b = 965.5$.
2. $A = 52° 41'$, $a = 55.71$.
3. $a = 0.2042$, $c = 0.2753$.
4. $A = 10° 51'$, $b = 7.123$.
5. $b = 5012$, $c = 8117$.
6. $A = 30° 18'$, $c = 0.02040$.
7. $B = 58° 15'$, $a = 48.04$.
8. $B = 6° 31'$, $b = 0.3691$.
9. $B = 23° 9'$, $b = 754.8$.
10. $A = 43° 49.2'$, $b = 22.568$.
11. $a = 2841.6$, $c = 6394.7$.
12. $A = 45° 11.6'$, $b = 61.496$.
13. $b = 862.35$, $c = 1036.0$.
14. $A = 14° 21.1'$, $c = 9.4726$.
15. $B = 26° 17.2'$, $a = 335.88$
16. $a = 0.18709$, $b = 0.22115$.
17. $B = 52° 9.8'$, $c = 73.211$.
18. $B = 34° 14.6'$, $b = 1202.2$.

19. (a) Find the base of an isosceles triangle whose vertex angle is $38° 27.2'$, and each of whose legs is 153.42. (b) Find the area of the triangle.

20. Find the side of a regular pentagon inscribed in a circle whose radius is 10.354 inches.

21. Find the radius of a circle in which a chord of 23.546 centimeters subtends an angle of $141° 18.4'$ at the center.

22. Find the area of a regular 5-pointed star inscribed in a circle of radius 12.517 inches.

Additional material for practice in the logarithmic solution of right triangles may be obtained from the exercises of Chapter II.

CHAPTER VI

Trigonometric Functions of Any Angle

33. Generation of an angle.

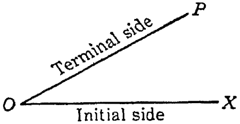


FIG. 30

The angle at O in Fig. 30 may be thought of as generated by the rotation of the line OP, from coincidence with OX to its present position. The line OX is called the **initial side** of the angle, OP is its **terminal side.**

34. Positive and negative angles.

It is evident that there is a choice of directions for rotating the generating line from the position OX to the position OP. One of these is that of the motion of the hands of a clock, and is called **clockwise,** the other is called **counterclockwise.** If the rotation of the generating line is counterclockwise, the angle is **positive** $(+)$; if the rotation is clockwise, the angle is **negative** $(-)$.* A small curved arrow, starting from the initial side and ending with its tip on the terminal side, is often used to indicate the direction of motion. (See Fig. 31.)

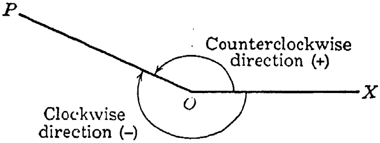


FIG. 31

It is evident that an angle may be of any magnitude

* There is no intrinsic reason why a counterclockwise rotation should give a positive angle and a clockwise rotation a negative angle. This designation, however, is the customary one.

(either positive or negative) whatever, for the generating line may rotate any number of times in either direction.

Any given position of OP represents an unlimited number of positive and negative angles.* On the other hand, to each angle, whether positive, negative, or zero, there corresponds one and only one position of OP.

Angles are equal if they are generated by the same amount of rotation in the same direction.

35. Rectangular coordinates.

Let us take two straight lines, OX and OY, intersecting at right angles at the point O. (See Fig. 32.) On each line we mark off a scale (same scale on each); positive numbers are to the right on the horizontal line OX, above on the vertical line OY; negative numbers are to the left on OX, below on OY. Line OX is called the **x-axis,** line OY the **y-axis,** point O the **origin.**

Now take any point P. The distance of the point from the y-axis is called the **abscissa** of the point and is denoted by x, its distance from the x-axis is called its **ordinate** and is denoted by y. The abscissa and ordinate together are called the **coordinates** (more specifically, **rectangular coordinates**) of the point. The point P in Fig. 32 has the abscissa -3 and the ordinate 4. For such a point it is customary to write $P(-3, 4)$, the abscissa being written first.

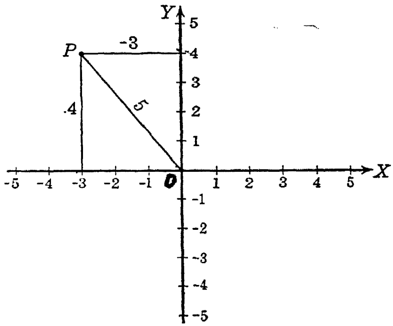


FIG. 32

Locating and marking the position of a point whose coordinates are given is called **plotting** the point.

* These angles may be called **coterminal,** since they have the same terminal side.

Besides the coordinates of the point, we find it convenient to consider its distance from the origin, which may be termed its **radius vector,** or simply its **radius,** and which we shall denote by r. Unless otherwise stated, r will for the present always be regarded as positive. (But see section 72.) Obviously we have $r^2 = x^2 + y^2$, and for the point P in the figure, $r = \sqrt{9 + 16} = 5$. Thus, for this particular point, we have $x = -3$, $y = 4$, $r = 5$.

36. Quadrants.

It will be noted that the coordinate axes divide the plane into four parts, called **quadrants,** numbered as shown in Fig. 33. The order of numbering is in accordance with counterclockwise rotation. That is, a line starting from coincidence with the positive end of the x-axis, and rotating about the origin O so as to generate a positive angle, turns first through quadrant I, then through quadrant II, and so on. Angles between 0° and 90° are in quadrant I, angles between 90° and 180° are in quadrant II, those between 180° and 270° are in quadrant III, those between 270° and 360° are in quadrant IV. Angles between 360° and 450° are in the first quadrant, and so on.

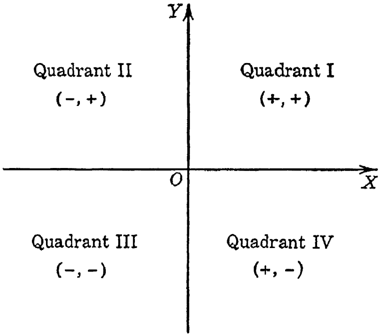


FIG. 33

The signs of x and y in each of the various quadrants are shown in Fig. 33 (the sign of x is written first) and in the following table:

Quadrant	I	II	III	IV
x (abscissa)	$+$	$-$	$-$	$+$
y (ordinate)	$+$	$+$	$-$	$-$

As already stated, the radius r will for the present be considered as always positive.

37. Trigonometric functions of any angle.

The definitions of the trigonometric functions given in section 2 suffice for acute angles only. In order to deal with the solution of oblique triangles and with other phases of trigonometry, it is necessary to generalize these definitions so that they will apply to any angle.

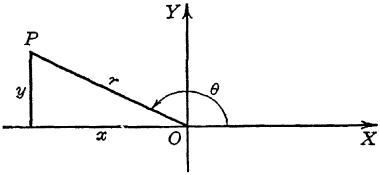


FIG. 34

To this end, let us consider the angle θ (Fig. 34), which has been generated by a line rotating about the origin, starting from coincidence with OX. Take any point P on its terminal side. With this point are associated three values: the abscissa x, the ordinate y, and the radius r. We define

$$\begin{aligned} \sin\theta &= \frac{\text{ordinate}}{\text{radius}} = \frac{y}{r}, & \csc\theta &= \frac{\text{radius}}{\text{ordinate}} = \frac{r}{y}, \\ \cos\theta &= \frac{\text{abscissa}}{\text{radius}} = \frac{x}{r}, & \sec\theta &= \frac{\text{radius}}{\text{abscissa}} = \frac{r}{x}, \qquad (1) \\ \tan\theta &= \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}, & \cot\theta &= \frac{\text{abscissa}}{\text{ordinate}} \quad \frac{x}{y}. \end{aligned}$$

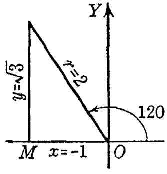


FIG. 35

These new definitions agree with those previously given (section 2) if the angle θ is in the first quadrant. As an illustration of their meanings for other angles, let us find the functions of 120°.

On the terminal side of an angle of 120°, whose initial side is the x-axis, take the point P so that $r = 2$. (See Fig. 35.) Then, angle $MOP = 60°$, and $x = -1$, from which we find, by

using the theorem of Pythagoras, that $y = \sqrt{3}$. The functions may now be read from the figure as follows:

$$\sin 120^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2},$$

$$\cos 120^\circ = \frac{x}{r} = \frac{-1}{2} = -\frac{1}{2},$$

$$\tan 120^\circ = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3},$$

$$\csc 120^\circ = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

$$\sec 120^\circ = \frac{r}{x} = \frac{2}{-1} = -2,$$

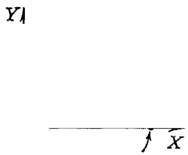
$$\cot 120^\circ = \frac{x}{y} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

EXERCISE

Show that the signs of the functions in the various quadrants are as shown in the following table.

quadrant	sin	cos	tan	csc	sec	cot
I	+	+	+	+	+	+
II	+	−	−	+	−	−
III	−	−	+	−	−	+
IV	−	+	−	−	+	−

38. Functions of 0°, 90°, 180°, 270°



$(x=1,\ y=0,\ r=1)$
FIG. 36

We may consider that we have an angle of 0° if there has been no rotation of the generating line. Take a point P on the terminal side of the angle, which of course coincides with the initial side, with any convenient abscissa, say 1. (See Fig. 36.) Then $x = 1$, $y = 0$, $r = 1$, and we have

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{1} = 0,$$

$$\cos 0^\circ = \frac{x}{r} = \frac{1}{1} = 1,$$

$$\tan 0^\circ = \frac{y}{x} = \frac{0}{1} = 0,$$

$$\csc 0^\circ = \frac{r}{y} = \frac{1}{0}, \text{ undefined,}$$

$$\sec 0^\circ = \frac{r}{x} = \frac{1}{1} = 1,$$

$$\cot 0^\circ = \frac{x}{y} = \frac{1}{0}, \text{ undefined.}$$

Note that csc 0° and cot 0° do not exist, since the ratios which would represent them have zero for denominator, and division by zero is impossible. However, as the angle θ shrinks to zero, cot θ * becomes numerically larger and larger without bound (e.g., cot $1' = 3437.7$, cot $1'' = 206265$). It is customary to express this fact by writing

$$\cot \theta \to \infty \text{ as } \theta \to 0, \tag{1}$$

where the symbol $\to$ is read " approaches " and the symbol ∞ is called **infinity.** The fact may also be written in the form

$$\lim_{\theta \to 0} \cot \theta = \infty, \tag{2}$$

which is read " the limit, as θ approaches zero, of cot θ is infinity." Either (1) or (2) is merely a shorthand notation for indicating that as the angle gets closer and closer to the value zero, the cotangent increases numerically without bound. It must be insisted that infinity (∞) is not a number.

* We select cot θ merely for purposes of illustration. A similar discussion holds for csc θ.

Similarly, from Fig. 37, in which each of the points P_1, P_2, P_3 is at a numerical distance of 1 from the origin, we can read off the functions of 90°, 180°, 270°. The values of these functions, as well as the functions of 0°, are tabulated below. The student should check them as an exercise. It is clear that the functions of 360° are the same as the functions of 0°. In the table the symbol ∞ is used to indicate that as the angle approaches the specified value, the corresponding function increases in numerical value without bound.

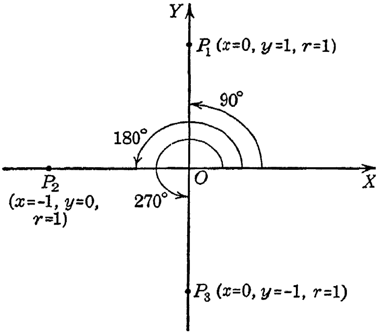


FIG. 37

angle	sin	cos	tan	csc	sec	cot
0°	0	1	0	∞	1	∞
90°	1	0	∞	1	∞	0
180°	0	-1	0	∞	-1	$-\infty$
270°	-1	0	∞	-1	$-\infty$	0

EXERCISES VI. A

Find the six functions of

1. 135°. **2.** 150°. **3.** 210°. **4.** 240°
5. 225°. **6.** 300°. **7.** 330°. **8.** 315°

Find the values of the following expressions:

9. $\sin 150° + \tan 225° + \cos 330°$.
10. $\cos 150° - 3 \tan 300° + 2 \sin 90°$.
11. $3 \tan 240° - \sin^2 135° + 2 \cot 210°$.
12. $3 \sin 135° + 2 \cos 225° - \tan 315°$.

13. $2 \cos 150° - 3 \sin 90° + \tan 210°$.
14. $(\cos 225° + \tan 45°)(\sin 135° + \cos 0°)$.
15. $(\tan 240° - \cos 300°)(2 \sin 300° + \frac{1}{2} \cot 225°)$.
16. $\sin^2 315° + \cos^2 270° + \tan^2 225°$.
17. $(\sin 315° + \cos 270° + \tan 225°)^2$.
18. $2 \cot 300° + 3 \cos 180° + \sin 270° \tan 150°$.
19. $\csc 150° + 2 \sec 330° + 5 \sin 180°$.
20. $3 \sec 135° - 2 \csc 225° + 4 \sin 315°$.
21. $\sec 150° \tan 300° + \tan 225° \csc^2 315°$.
22. $(5 \cos 270° + \sec 180° - \frac{1}{3} \sin 360°)^3$.
23. $(\frac{1}{2} \sec 240° + \csc^2 315° - \cot 135°)^2$.
24. $\sqrt{2} \tan 135° + \sqrt{3} \sin 240° + \sqrt{5} \csc 270°$.
25. $\dfrac{\cos 300° + \cos 360°}{\sin 150° + \sec 300°}$.
26. $\dfrac{3 \tan 135° + 2 \cos 225°}{\sin 240° + \tan 300°}$.
27. $\dfrac{\cot 225° + \sin 270°}{\sec 225° - \tan 300°}$.

39. Functions of $-\theta$.

Let us consider the functions of $-\theta$, where θ is any angle whatever. In Fig. 38 the angle θ is, for definiteness, shown in the first quadrant, but in the following considerations θ is not restricted to the first, or to any other quadrant. It is readily seen that in the congruent right triangles OMP' and OMP, $x' = x$, $y' = -y$ (since MP' and MP extend in opposite directions), and $r' = r$ (since the radius is to be regarded as positive). Consequently,

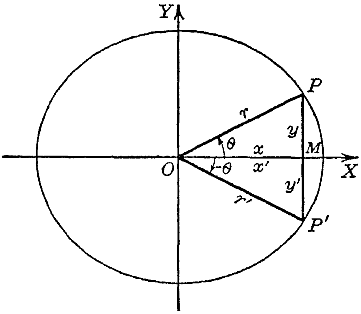


FIG. 38

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin \theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta,$$

$$\csc(-\theta) = \frac{r'}{y'} = \frac{r}{-y} = -\frac{r}{y} = -\csc\theta,$$

$$\sec(-\theta) = \frac{r'}{x'} = \frac{r}{x} = \sec\theta,$$

$$\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\frac{x}{y} = -\cot\theta.$$

EXERCISE

Prove the formulas of section 39 by means of a figure in which θ is an angle in (a) quadrant II, (b) quadrant III, (c) quadrant IV.

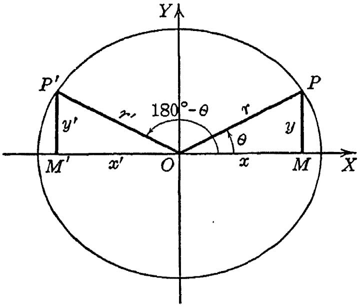


Fig. 39

40. Functions of 180° − *θ*.

Let us now consider the functions of $180^\circ - \theta$, where again θ may be any angle whatever. Reference to Fig. 39, in which $OM'P'$ and OMP are congruent right triangles, shows that

$$\sin(180^\circ - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin\theta,$$

$$\cos(180^\circ - \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta,$$

$$\tan(180^\circ - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\frac{y}{x} = -\tan\theta,$$

$$\csc(180^\circ - \theta) = \frac{r'}{y'} = \frac{r}{y} = \csc\theta,$$

$$\sec(180^\circ - \theta) = \frac{r'}{x'} = \frac{r}{-x} = -\frac{r}{x} = -\sec\theta,$$

$$\cot(180^\circ - \theta) = \frac{x'}{y'} = \frac{-x}{y} = -\frac{x}{y} = -\cot\theta.$$

EXERCISE

Prove the formulas of section 40 by means of a figure in which θ is an angle in (a) quadrant II, (b) quadrant III, (c) quadrant IV.

41. Functions of 180° + θ.

By the same method of proof, it can be shown from Fig. 40, that

$$\begin{aligned}
\sin(180^\circ + \theta) &= -\sin\theta,\\
\csc(180^\circ + \theta) &= -\csc\theta,\\
\cos(180^\circ + \theta) &= -\cos\theta,\\
\sec(180^\circ + \theta) &= -\sec\theta,\\
\tan(180^\circ + \theta) &= \tan\theta,\\
\cot(180^\circ + \theta) &= \cot\theta.
\end{aligned}$$

This is left as an exercise for the student.

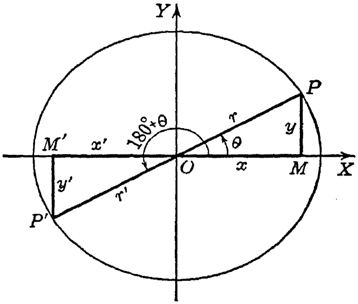


FIG. 40

42. Functions of 360° − θ.

From Fig. 38, it is evident that the functions of $360^\circ - \theta$ are the same as the functions of $-\theta$. Thus,

$$\begin{aligned}
\sin(360^\circ - \theta) &= -\sin\theta, & \csc(360^\circ - \theta) &= -\csc\theta,\\
\cos(360^\circ - \theta) &= \cos\theta, & \sec(360^\circ - \theta) &= \sec\theta,\\
\tan(360^\circ - \theta) &= -\tan\theta, & \cot(360^\circ - \theta) &= -\cot\theta.
\end{aligned}$$

43. Functions of 360° + θ.

It should be quite clear that the functions of $360^\circ + \theta$ are the same as the corresponding functions of θ, since these two angles are coterminal. (See footnote, page 65.)

44. Functions of $90° - \theta$.

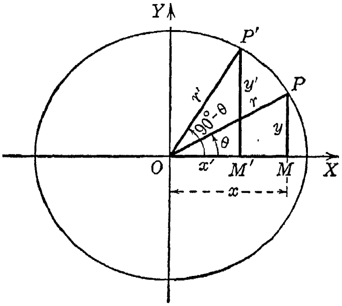


FIG. 41

It was shown in section 3 that, for any acute angle A, $\sin(90° - A) = \cos A$, etc. That is, any function of an acute angle is equal to the cofunction of the complementary angle. That formulas (2) of section 3 are true for any angle may be shown by Fig. 41 as follows:

Right triangles $OM'P'$ and OMP are congruent, and consequently $x' = y$, $y' = x$, $r' = r$. Therefore,

$$\sin(90° - \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° \quad \theta) = \frac{x'}{r'} = \frac{y}{r} = \sin\theta,$$

$$\tan(90° - \theta) = \frac{y'}{x'} = \frac{x}{y} = \cot\theta,$$

$$\csc(90° - \theta) = \frac{r'}{y'} = \frac{r}{x} = \sec\theta,$$

$$\sec(90° - \theta) = \frac{r'}{x'} = \frac{r}{y} = \csc\theta,$$

$$\cot(90° - \theta) = \frac{x'}{y'} = \frac{y}{x} = \tan\theta.$$

EXERCISE

Prove the formulas of section 44 by means of a figure in which θ is an angle in (a) quadrant II, (b) quadrant III, (c) quadrant IV.

45. Functions of $90° + \theta$. omit

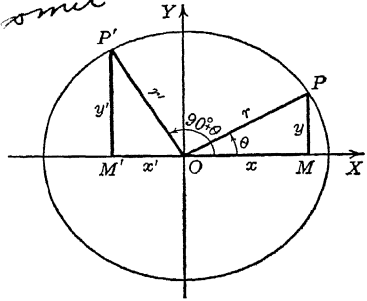


Fig. 42

It is seen that in Fig. 42, x' and y are numerically equal but have opposite signs; that is, $x' = -y$. Similarly, y' and x are numerically equal and have the same sign; that is, $y' = x$. Also, $r' = r$. It follows that

$$\sin(90° + \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° + \theta) = \frac{x'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta,$$

$$\tan(90° + \theta) = \frac{y'}{x'} = \frac{x}{-y} = -\frac{x}{y} = -\cot\theta,$$

$$\csc(90° + \theta) = \frac{r'}{y'} = \frac{r}{x} = \sec\theta,$$

$$\sec(90° + \theta) = \frac{r'}{x'} = \frac{r}{-y} = -\frac{r}{y} = -\csc\theta,$$

$$\cot(90° + \theta) = \frac{x'}{y'} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta.$$

EXERCISE

Prove the formulas of section 45 by means of a figure in which θ is an angle in (a) quadrant II, (b) quadrant III, (c) quadrant IV.

46. Functions of $270° - \theta$. omit

In Fig. 43, $x' = -y$, $y' = -x$, $r' = r$, and it can readily be proved that

$\sin(270° - \theta) = -\cos\theta,$ $\qquad$ $\csc(270° - \theta) = -\sec\theta,$

$$\cos(270° - \theta) = -\sin\theta, \qquad \sec(270° - \theta) = -\csc\theta,$$
$$\tan(270° - \theta) = \cot\theta, \qquad \cot(270° - \theta) = \tan\theta.$$

Proofs are left as exercises for the student.

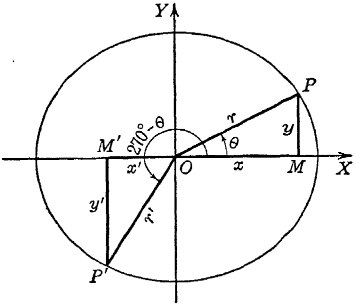


FIG. 43

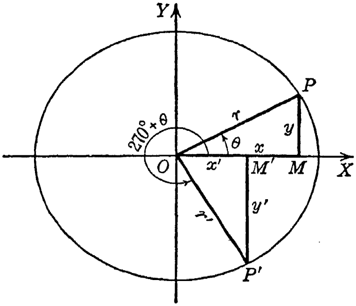


FIG. 44

47. Functions of 270° + θ.

omit

In Fig. 44, $x' = y$, $y' = -x$, $r' = r$, and it follows that

$$\sin(270° + \theta) = -\cos\theta, \qquad \csc(270° + \theta) = -\sec\theta,$$
$$\cos(270° + \theta) = \sin\theta, \qquad \sec(270° + \theta) = \csc\theta,$$
$$\tan(270° + \theta) = -\cot\theta, \qquad \cot(270° + \theta) = -\tan\theta.$$

Proofs are left as exercises.

48. Summary.

The formulas of sections 39–47 may be summarized as in the accompanying table. The upper sign preceding a function corresponds to the upper sign in the angle at the left of the same row, and similarly for the lower sign.

angle	sin	cos	tan	csc	sec	cot
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\csc\theta$	$\sec\theta$	$-\cot\theta$
$90° \pm \theta$	$\cos\theta$	$\mp\sin\theta$	$\mp\cot\theta$	$\sec\theta$	$\mp\csc\theta$	$\mp\tan\theta$
$180° \pm \theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\tan\theta$	$\mp\csc\theta$	$-\sec\theta$	$\pm\cot\theta$
$270° \pm \theta$	$-\cos\theta$	$\pm\sin\theta$	$\mp\cot\theta$	$-\sec\theta$	$\pm\csc\theta$	$\mp\tan\theta$
$360° \pm \theta$	$\pm\sin\theta$	$\cos\theta$	$\pm\tan\theta$	$\pm\csc\theta$	$\sec\theta$	$\pm\cot\theta$

Note that in any column we have the same function as that at the head of the column, except for the rows having $90° \pm \theta$ and $270° \pm \theta$ at the left; in these rows we find the cofunctions.

The student should make no attempt to memorize this table, but he should be able to work out any of the results listed in it by the methods of the preceding sections; that is, by drawing a figure for each separate problem as needed.

For the important special case in which θ is an acute angle the following statements may prove helpful: If an angle is written in the form $-\theta$, $180° \pm \theta$, or $360° \pm \theta$ we may say that it is **referred to the x-axis;** if it is written in the form $90° \pm \theta$ or $270° \pm \theta$, we may say that it is **referred to the y-axis;** in either case we shall call θ the **reference angle.** *The function of any angle referred to the x-axis is numerically equal to the same function of the reference angle; the function of any angle referred to the y-axis is numerically equal to the cofunction of the reference angle. The sign to be prefixed to the resulting function of θ is that of the original function, as determined by the quadrant in which the original angle is siti*

49. Reduction of functions of any angle to functions of an acute angle.

We are now in a position to find the functions of any angle whatever.

Example 1.

Find sine, cosine, and tangent of 110°.

SOLUTION. Since $110° = 180° - 70°$, we have

$$\begin{aligned}
\sin 110° &= \sin(180° - 70°) = \sin 70° = 0.9397,\\
\cos 110° &= \cos(180° - 70°) = -\cos 70° = -0.3420,\\
\tan 110° &= \tan(180° - 70°) = -\tan 70° = -2.7475.
\end{aligned}$$

Or, since $110^\circ = 90^\circ + 20^\circ$,

$$\begin{aligned}
\sin 110^\circ &= \sin(90^\circ + 20^\circ) = \cos 20^\circ = 0.9397,\\
\cos 110^\circ &= \cos(90^\circ + 20^\circ) = -\sin 20^\circ = -0.3420,\\
\tan 110^\circ &= \tan(90^\circ + 20^\circ) = -\cot 20^\circ = -2.7475.
\end{aligned}$$

Example 2.

Find sine, cosine, and tangent of 615°.

Solution. Since $615^\circ = 360^\circ + 255^\circ$, the functions of 615° are exactly the same as those of 255°. But $255^\circ = 180^\circ + 75^\circ$. Thus,

$$\begin{aligned}
\sin 615^\circ &= \sin 255^\circ = \sin(180^\circ + 75^\circ) = -\sin 75^\circ = -0.9659,\\
\cos 615^\circ &= \cos 255^\circ = \cos(180^\circ + 75^\circ) = -\cos 75^\circ = -0.2588,\\
\tan 615^\circ &= \tan 255^\circ = \tan(180^\circ + 75^\circ) = \tan 75^\circ = 3.7321.
\end{aligned}$$

Or, we could express 255° as $270^\circ - 15^\circ$.

EXERCISES VI. B

1. Express each of the following as a function of a positive acute angle:

(a) $\sin 160^\circ$, (b) $\cos 145^\circ$, (c) $\tan 100^\circ$,
(d) $\csc 130^\circ$, (e) $\sec 172^\circ$, (f) $\cot 98^\circ$,
(g) $\sin 137^\circ$, (h) $\cos 95^\circ 10'$, (i) $\tan 162^\circ 4'$,
(j) $\cot 125^\circ 18'$, (k) $\sin 114^\circ 21'$, (l) $\cos 92^\circ 12.8'$.

2. Reduce each of the following to a function of a positive angle less than 45°:

(a) $\sin 175^\circ$, (b) $\cos(-167^\circ)$, (c) $\tan 520^\circ$,
(d) $\cot 125^\circ 26'$, (e) $\sec 267^\circ 28'$, (f) $\csc 325^\circ 41.8'$,
(g) $\sin 215^\circ 5'$, (h) $\cos 281^\circ 22'$, (i) $\tan 197^\circ 35'$,
(j) $\cot 312^\circ 54'$, (k) $\sin 356^\circ 56'$, (l) $\cos 95^\circ 6.5'$.

3. Find the numerical value of

(a) $\sin 145^\circ$, (b) $\cos 246^\circ$, (c) $\tan 285^\circ$,
(d) $\cot 572^\circ 38'$, (e) $\cos 321^\circ$, (f) $\sin 642^\circ 50.5'$,
(g) $\cot 121^\circ 13.6'$, (h) $\sin 462^\circ 31.1'$, (i) $\sin(-162^\circ 45')$,
(j) $\cos(-72^\circ 15')$, (k) $\tan(-200^\circ)$, (l) $\cot(-275^\circ 18')$.

Find the value of

4. $\cos 240° \cos 120° - \sin 120° \cos 150°$.
5. $\tan 315° \sec 900° + \cot 495° \csc 450°$.
6. $\sin(90° + \theta) \sin(180° + \theta) + \cos(90° + \theta) \cos(180° - \theta)$.
7. Given that θ is the angle of a triangle, find θ if

   (a) $\sin \theta = 0.3090$, (b) $\cos \theta = 0.4975$, (c) $\tan \theta = 2.8770$,
   (d) $\cot \theta = 1.7090$, (e) $\sin \theta = 0.6713$, (f) $\cos \theta = -0.7716$.

8. Express as functions of θ:

   (a) $\sin(810° - \theta)$, (b) $\tan(990° - \theta)$, (c) $\cot(\theta - 360°)$,
   (d) $\sec(\theta - 90°)$, (e) $\cos(-180° - \theta)$, (f) $\csc(630° + \theta)$.

CHAPTER VII

Solution of Oblique Triangles

50. The four cases.

We shall now take up the solution of oblique triangles by methods that do not require breaking them up into right triangles, as was done in section 11. Problems in the solution of oblique triangles may be classified into the following four cases, already mentioned in that section:

Case I. Two angles and a side given.

Case II. Two sides and the angle opposite one of them given.

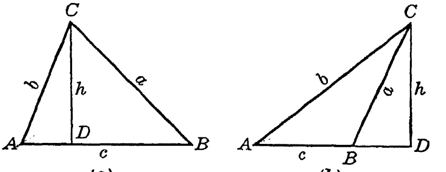
Case III. Two sides and the included angle given.

Case IV. Three sides given.

Certain formulas are necessary for handling the various cases, and these will be developed as needed.

51. Law of sines.

Fig. 45(a) represents an acute triangle, Fig. 45(b) an ob-



(a) (b)

FIG. 45

tuse triangle, B being the obtuse angle. In each figure we draw the altitude CD and designate its length by h. Then, in Fig. 45(a),

$$\sin B = \frac{h}{a}, \qquad \text{or} \qquad h = a \sin B, \tag{1}$$

and the same relation holds in Fig. 45(b), since

$$\sin(180^\circ - B) = \sin B.$$

In either figure,

$$\sin A = \frac{h}{b}, \qquad \text{or} \qquad h = b \sin A. \tag{2}$$

Equating the values of h in (1) and (2), we have

$$a \sin B = b \sin A, \tag{3}$$

and dividing both sides of (3) by $\sin A \sin B$, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{4}$$

Similarly, by drawing the altitude from A, we can prove that

$$\frac{}{\sin B} = \frac{}{\sin C} \tag{5}$$

Combining (4) and (5), we obtain the **law of sines,**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \tag{6}$$

which may be stated in words as follows: *The sides of a triangle are proportional to the sines of the opposite angles.*

EXERCISE

Prove that if $C = 90^\circ$, formula (6) reduces to the definitions of $\sin A$ and $\sin B$.

A formula for the area of a triangle is easily derivable from formula (2) for the altitude. Since the area is equal

to half the product of the base and the altitude, we have

$$\text{area} = \frac{1}{2}\, bc \sin A. \tag{7}$$

The area is also of course equal to $\frac{1}{2}\, ac \sin B$ and $\frac{1}{2}\, ab \sin C$. In words, *the area of a triangle is equal to one-half the product of any two sides and the sine of the included angle.*

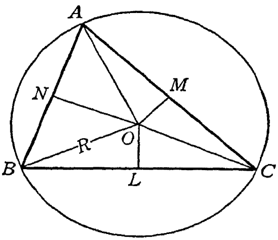


Fig. 46

The following proof of the law of sines gives a geometric meaning to the equal ratios in (6):

Draw the perpendicular bisectors of the sides of the triangle ABC (Fig. 46). They will meet in a point O, which is the center of the circumscribed circle. Draw this circle, and connect its center with the vertices of the triangle. Let R be the radius of the circle, and, as usual, let A, B, C represent the angles of the triangle.

Then, angle $BOC = 2A$. (Why?)
Hence, angle $BOL = A$.
Consequently,

$$\sin A = \sin BOL = \frac{BL}{R} = \frac{\frac{1}{2}a}{R} = \frac{a}{2R}.$$

Similarly,

$$\sin B = \frac{b}{2R}, \qquad \sin C = \frac{c}{2R},$$

and it follows that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = D, \tag{8}$$

where D is the diameter of the circumscribed circle.

If one of the angles of the triangle is obtuse, the proof requires a slight modification.

52. Solution of Case I.

This case, in which there are *two angles and a side given,* can be solved by the law of sines.

Example.

Solve the triangle $A = 40°$, $B = 60°$, $c = 50$.

Solution. $C = 180° - (A + B) = 80°$.
From the law of sines,

$$a = \frac{c \sin A}{\sin C} = \frac{50 \sin 40°}{\sin 80°} = \frac{50 \times 0.6428}{0.9848} = 32.6,$$

$$b = \frac{c \sin B}{\sin C} = \frac{50 \sin 60°}{\sin 80°} = \frac{50 \times 0.8660}{0.9848} = 44.0.$$

These results may be checked by using the relation $a/\sin A = b/\sin B$, or by means of **Mollweide's equation,**

$$\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}, \tag{1}$$

which is proved in section 61. (If $B > A$, interchange A and B, a and b, respectively, in the formula.)

They may also be checked by using one of the following relations, proofs of which are left as exercises:

$$a = b \cos C + c \cos B, \qquad b = a \cos C + c \cos A, \qquad c = a \cos B + b \cos A. \tag{2}$$

EXERCISES VII. A

Solve the following triangles:

1. $A = 70°$, $B = 80°$, $a = 12$.
2. $A = 70°$, $B = 80°$, $c = 12$.
3. $A = 58° 10'$, $C = 84° 40'$, $b = 2.5$.
4. $B = 132° 10'$, $C = 18° 20'$, $c = 10.2$.
5. $B = 10° 50'$, $C = 75° 30'$, $b = 61$.
6. $A = 95° 40'$, $C = 45° 20'$, $a = 8.2$.

✓7. The bases of a trapezoid are 22 and 12 respectively. The angles at the extremities of one base are 65° and 40° respectively. Find the two legs.

8. Two observers, who are 2 miles apart on a horizontal plane, observe a balloon in the same vertical plane with themselves. The angles of elevation are 50° and 65° respectively. Find the height of the balloon, (a) if it is between the observers; (b) if it is on the same side of both of them. ✓

9. One diagonal of a parallelogram is 16.5. It makes angles of 36° 10′ and 14° 30′ respectively with the sides. Find the sides.

10. A line AB, 125 feet long, is measured along the straight bank of a river. A point C is on the opposite bank. Angles ABC and BAC are found to be 65° 40′ and 54° 30′ respectively. How wide is the river?

11. From a certain point the angle of elevation of the top of a building is 38°. From a point 75 feet nearer the building the angle of elevation is 65°. Find the height of the building.

12. From a given position an observer notes that the angle of elevation of a rock is 47°. After walking 1000 feet towards the rock, up a slope of 32°, he finds the angle of elevation to be 75°. Find the vertical distance of the rock above each point of observation.

13. A flagpole 25 feet tall stands on top of a building. From a point in the same horizontal plane with the base of the building the angles of elevation of the top and the bottom of the flagpole are 61° 30′ and 56° 20′ respectively. How high is the building?

14. Find the radius of the circle circumscribed about the triangle for which $A = 50°$, $B = 20°$, $a = 35$.

53. Solution of Case II.

This case, in which we have *two sides and the angle opposite one of them given*, presents difficulties that are not found in the other cases. This is because we sometimes find two solutions for the problem; that is, we find two triangles having the given parts. Sometimes we find only one triangle, and sometimes, indeed, we do not find any; that

is, the problem is impossible. A carefully constructed figure will usually show how many solutions there are, but the following discussion explains how this can be determined accurately:

Let us suppose that the given parts are a, b, A.

We consider first the case in which A is acute. Construct this angle, and mark off the point C on one of its sides so that $AC = b$. Extend the other side indefinitely. (See Fig. 47.)

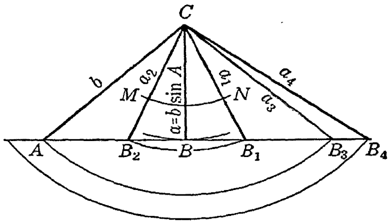


FIG. 47

The perpendicular distance from C to this extended side is $b \sin A$, and it is evident that various cases may occur, depending upon the length of a as compared with b and with $b \sin A$.

Let us take a pair of compasses, and with C as center and a as radius, test these various cases by constructing arcs.

If a is less than $b \sin A$, the arc will be like MN, and there will be no triangle.

If $a = b \sin A$, the arc will be tangent to the base line (that is, the extended side) at the point B, and there will be but one triangle, the right triangle ABC.

If a is greater than $b \sin A$ but less than b, the arc will cut the base line in two points, such as B_1 and B_2. Consequently, we get two triangles, AB_1C and AB_2C. Under these conditions, Case II is said to be **ambiguous**, that is, there is not a unique solution. Since either of the triangles satisfies the requirements of the problem, we must solve both.

If $a = b$, the arc passes through A, and we get but one solution, the isosceles triangle AB_3C.

If a is greater than b, there is but one triangle, such as AB_4C.

There are no other possible conditions when A is acute.

If A is a right angle, as shown in Fig. 48, it is evident that we cannot have a triangle unless a is greater than b, under which condition we have only one construction.

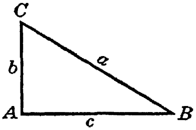


FIG. 48

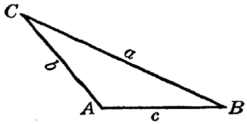


FIG. 49

If A is obtuse, as in Fig. 49, the arc having a as radius cannot cut the base line on the proper side of the point A unless a is greater than b. Thus, we have no triangle unless a is greater than b, and then we have only one.

Our conclusions may be summarized as follows:

$A < 90^\circ$	
$a < b \sin A$	no solution
$a = b \sin A$	one solution (right triangle)
$b \sin A < a < b$	two solutions
$a = b$	one solution (isosceles triangle)
$a > b$	one solution

$A \geqq 90^\circ$	
$a \leqq b$	no solution
$a > b$	one solution

If the given parts are other than a, b, A, the foregoing summary must, of course, be modified accordingly.

Case II is solved by the application of the law of sines.

Example.

Solve the triangle $a = 20$, $b = 10$, $A = 75^\circ$.

SOLUTION. It is apparent here that there is only one solution. From the law of sines, we have

$$\sin B = \frac{b \sin A}{a} = \frac{10 \sin 75^\circ}{20} = \frac{10 \times 0.9659}{20} = 0.4830,$$

$$B = 28^\circ\ 50',$$

$$C = 180^\circ - (A + B) = 180^\circ \quad 103^\circ\ 50' = 76^\circ\ 10',$$

$$c = \frac{a \sin C}{\sin A} = \frac{20 \sin 76^\circ 10'}{\sin 75^\circ} = \frac{20 \times 0.9710}{0.9659} = 20.1.$$

The results may be checked by computing c from the relation $c = b \sin C/\sin B$, or by using Mollweide's equation (1) of the preceding section.

Note that from the value $\sin B = 0.4830$ we could also have $B = 180^\circ - 28^\circ 50' = 151^\circ 10'$. However, if we should attempt to find C by adding A and B and subtracting their sum from 180°, we should find $A + B = 75^\circ + 151^\circ 10' = 226^\circ 10'$, which is impossible. This method will always show whether there is a second solution.

EXERCISES VII. B

Solve the following triangles:

1. $A = 40^\circ$,	$a = 8$,	$b = 5$.
2. $A = 30^\circ$,	$a = 5$,	$b = 8$.
3. $B = 36^\circ 10'$,	$a = 21.2$,	$b = 31.0$.
4. $C = 108^\circ 20'$,	$b = 12.2$,	$c = 25.1$.
5. $A = 73^\circ 20'$,	$a = 2.5$,	$b = 1.8$.
6. $B = 30^\circ$,	$b = 99$,	$a = 198$.
7. $C = 15^\circ 40'$,	$a = 35$,	$c = 9.5$.
8. $B = 65^\circ 30'$,	$a = 17.6$,	$b = 15.9$.

9. A side and a diagonal of a parallelogram are 12 inches and 19 inches respectively. The angle between the diagonals, opposite the given side, is 124°. Find the length of the other diagonal and the length of the other side.

10. A lighthouse is 10 miles northeast of a dock. A ship leaves the dock at noon, and sails east at a speed of 12 miles an hour. At what time will it be 8 miles from the lighthouse?

11. A vertical pole 35 feet high, standing on sloping ground, is braced by a wire which extends from the top of the pole to a point on the ground 25 feet from the foot of the pole. If the pole subtends an angle of 30° at the point where the wire reaches the ground, how long is the wire?

12. A tower 125 feet high stands on the side of a hill. At a point 240 feet from the foot of the tower, measured straight down the hill, the tower subtends an angle of 25°. What angle does the side of the hill make with the horizontal?

54. Law of cosines.

In Fig. 50(a), angle A is acute; in Fig. 50(b), angle A is obtuse. In each figure let us draw the altitude CD, whose numerical value we set equal to h. Further, let $AD = m$. Then, in Fig. 50(a),

$$a^2 = h^2 + (c - m)^2 = h^2 + c^2 - 2\,cm + m^2, \tag{1}$$

while in Fig. 50(b),

$$a^2 = h^2 + (c + m)^2 = h^2 + c^2 + 2\,cm + m^2. \tag{2}$$

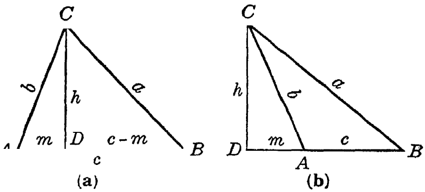


FIG. 50

Since, in either figure, $h^2 + m^2 = b^2$, (1) and (2) reduce respectively to

$$a^2 = b^2 + c^2 - 2cm, \tag{3}$$

and

$$a^2 = b^2 + c^2 + 2cm. \tag{4}$$

But in Fig. 50(a),

$$m = b \cos A,$$

and in Fig. 50(b),

$$m = b \cos(180° - A) = -b \cos A.$$

Substituting these values of m in (3) and (4) respectively, we obtain

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A.} \tag{5}$$

Similarly, $$\mathbf{b^2 = c^2 + a^2 - 2ca \cos B,} \tag{6}$$

and $$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C.} \tag{7}$$

These three formulas constitute the **law of cosines,** which states that *the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these two sides times the cosine of the angle between them.*

NOTE. The law of cosines combines into one statement the following three theorems of plane geometry:

I. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two sides.

II. In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of either of those sides by the projection of the other upon it.

III. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides by the projection of the other upon it.

Formulas (6) and (7) may be obtained from (5) by what is termed a **cyclic** change of letters. This may be effected in the following way:

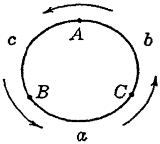


FIG. 51

Arrange the letters around the circumference of a circle, as in Fig. 51. Then replace each letter in the given formula by the next in order. Thus, a new formula is obtained if

$$\boldsymbol{a} \textit{ is replaced by } \boldsymbol{b},$$
$$\boldsymbol{b} \textit{ is replaced by } \boldsymbol{c},$$
$$\boldsymbol{c} \textit{ is replaced by } \boldsymbol{a},$$

and similarly for the capital letters.

In this manner (5) is changed into (6), which in turn may be changed into (7).

Note that if C is a right angle, (7) becomes the Pythagorean relation, $c^2 = a^2 + b^2$, since $\cos 90° = 0$.

EXERCISE

Show that if $C = 90°$, (5) and (6) reduce to the definitions of $\cos A$ and $\cos B$ respectively.

55. Solution of Case III.

The law of cosines is useful in solving Case III, in which we have *two sides and the included angle given.*

Example.

Solve the triangle $a = 25$, $b = 30$, $C = 50°$.

SOLUTION.
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ &= (25)^2 + (30)^2 - 2 \times 25 \times 30 \times \cos 50° \\ &= 625 + 900 - 1500 \times 0.6428 = 560.8, \\ c &= 23.7. \end{aligned}$$

Angles A and B may be found by the law of sines.

The smaller of these angles should be found first, for if the larger is obtuse some confusion may arise.

A check is afforded by Mollweide's equation (1) of section **52**.

EXERCISES VII. C

Solve the following triangles:

✓**1.** $a = 5,\quad c = 6,\quad B = 60°.$
2. $a = 2,\quad b = 3,\quad C = 130°.$
3. $b = 1.7,\quad c = 2.2,\quad A = 17°\,20'.$
4. $a = 0.35,\quad b = 0.24,\quad C = 75°\,40'.$
5. $a = 230,\quad b = 150,\quad C = 95°.$
6. $b = 80.1,\quad c = 106,\quad A = 165°\,50'.$
✓**7.** Two ships leave a dock at the same time. One sails northeast at the rate of 8.5 miles an hour, the other sails north at the rate of 10 miles an hour. How far apart are they at the end of 2 hours?
8. If the slower ship in the preceding exercise leaves at noon, and the other at 1 p.m., how far apart are they at 2 p.m.?
✓**9.** The diagonals of a parallelogram are 7 inches and 9 inches respectively; they intersect at an angle of 52°. Find the sides of the parallelogram.
10. A military observer notes two enemy batteries which subtend, at his observation post, an angle of 40°. The interval between the flash and the report of a gun is 5 seconds for one battery, and 4 seconds for the other. If the velocity of sound is 1140 feet a second, how far apart are the batteries?
11. Points A and B are separated by an obstacle. In order to find the distance between them, a third point C is selected which is 120 yards from A and 150 yards from B. The angle

ACB is measured to be 80° 10′. Find the distance from A to B.

12. Two circles, whose radii are 12 inches and 16 inches respectively, intersect. The angle between the tangents at either of the points of intersection is 29° 30′. Find the distance between the centers of the circles.

56. Solution of Case IV.

Case IV, *three sides given,* can also be solved by the law of cosines.

Example.

Solve the triangle $a = 5$, $b = 6$, $c = 9$.

SOLUTION. Solving the law of cosines $a^2 = b^2 + c^2 - 2bc \cos A$ for $\cos A$, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \frac{36 + 81 - 25}{2 \times 6 \times 9} = \frac{92}{108} = 0.8519,$$

$$A = 31° \, 35'.$$

Similarly,

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \quad \frac{81 + 25 - 36}{2 \times 9 \times 5} = \frac{70}{90} = 0.7778,$$

$$B = 38° \, 57';$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \frac{25 + 36 - 81}{2 \times 5 \times 6} = -\frac{20}{60} = -0.3333,$$

$$C = 180° - 70° \, 32' = 109° \, 28'.$$

CHECK. $A + B + C = 180°$.

EXERCISES VII. D

Find the angles of the following triangles:

1. $a = 2$, $b = 3$, $c = 4$.
2. $a = 0.013$, $b = 0.014$, $c = 0.015$.
3. $a = 8.4$, $b = 7.2$, $c = 6.5$.
4. $a = 45$, $b = 32$, $c = 71$.
5. $a = 1.4$, $b = 4.8$, $c = 5.0$.
6. $a = 24$, $b = 7$, $c = 25$.
7. $a = 13.2$, $b = 11.8$, $c = 20.1$.
8. $a = 20.1$, $b = 21.0$, $c = 15.5$.

9. Three towns, A, B, and C, are situated so that $AB = 300$ miles, $AC = 194$ miles, and $BC = 160$ miles, B being due north of C. Find the direction from B to A.

10. A ladder 20 feet long is set with one end at a horizontal distance of 7 feet from a sloping wall. The other end of the ladder reaches 15 feet up the face of the wall. What angle does the wall make with the horizontal?

11. The sides of a parallelogram are 11.7 inches and 15.0 inches respectively; one diagonal is 13.1 inches. Find the angles. Also find the other diagonal.

12. If the sides of a triangle are 16, 20, and 27 respectively, what is the length of the bisector of the largest angle?

13. Find the length of the median to the longest side in the preceding exercise.

14. Three circles of radii 3, 4, and 5 inches respectively are tangent to each other externally. Find the angles of the triangle formed by joining the centers.

Omit

★57. Application of law of cosines to Case II.

It may be noted that Case II can be handled by the law of cosines.

Example.

Solve the triangle $a = 20$, $b = 10$, $A = 75°$.

SOLUTION. Substitute the given values in the equation

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This gives
$$\begin{aligned} 400 &= 100 + c^2 - 2 \times 10 \times c \times \cos 75° \\ &= 100 + c^2 - 20c \times 0.2588, \end{aligned}$$

which reduces to the quadratic equation

$$c^2 - 5.176c - 300 = 0.$$

$$c = \frac{5.176 \pm \sqrt{(5.176)^2 + 1200}}{2} = \frac{5.176 \pm 35.026}{2} = 20.1.$$

There is also a negative root of the equation, but it is discarded. If there are two positive roots, it means that there are two solutions.

The method is particularly useful if it is not required to find the remaining two angles. However, if they are required, they may be found either by the law of sines or by the law of cosines.

EXERCISE

Solve, by using the law of cosines, exercise VII. B, 10; also such other exercises of VII. B as the instructor may assign.

58. Logarithmic solution of Case I.

The solution of this case by logarithms follows the same steps as the solution in section 52. The only difference is that logarithms are employed in performing the computations.

Example.

Solve the triangle $A = 79° \, 59.3'$, $B = 46° \, 36.4'$, $a = 804.32$.

SOLUTION.

$$C = 180° - (A + B).$$

$$b = \frac{a \sin B}{\sin A},$$

$$\log b = \log a + \log \sin B + \operatorname{colog} \sin A.$$

$$c = \frac{a \sin C}{\sin A},$$

$$\log c = \log a + \log \sin C + \operatorname{colog} \sin A.$$

A	$79° \, 59.3'$
B	$46° \, 36.4'$
$A + B$	$126° \, 35.7'$
C	**$53° \, 24.3'$**
a	804.32
$\log \sin B$	9.86133 − 10
$\log a$	2.90543
$\operatorname{colog} \sin A$	0.00666
$\log \sin C$	9.90465 − 10
$\log b$	2.77342
$\log c$	2.81674
b	**593.50**
c	**655.75**

CHECK.

$$\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C} = x,$$

$$\log x = \log(a + b) - \log c,$$

$$\log x = \log \cos \tfrac{1}{2}(A - B) - \log \sin \tfrac{1}{2}C.$$

$a + b$	1397.82
$A - B$	$33° \, 22.9'$
$\frac{1}{2}(A - B)$	$16° \, 41.45'$
$\frac{1}{2}C$	$26° \, 42.15'$
$\log(a + b)$	3.14545
$\log c$	2.81674
$\log x$	0.32871
$\log \cos \frac{1}{2}(A - B)$	9.98131 − 10
$\log \sin \frac{1}{2}C$	9.65259 − 10
$\log x$	0.32872

It should be noted that, in checking, we do not need to find the quantities $(a + b)/c$ and $\cos \frac{1}{2}(A - B)/\sin \frac{1}{2}C$; it is sufficient if the logarithms of these quantities agree. Slight discrepancies in the last place are to be expected.

EXERCISES VII. E

Find the remaining parts, and also the areas, of the following triangles:

1.	$B = 65° 25.5'$,	$C = 81° 24.6'$,	$b = 724.32$.
2.	$B = 38° 37.4'$,	$C = 75° 32.8'$,	$c = 129.63$.
3.	$A = 48° 29.2'$,	$C = 115° 33.8'$,	$a = 14.829$.
4.	$A = 68° 41.5'$,	$C = 110° 16.5'$,	$c = 9.4326$.
5.	$A = 11° 11.3'$,	$C = 57° 37.4'$,	$c = 444.79$.
6.	$B = 20° 20.2'$,	$C = 12° 28.5'$,	$a = 673.75$.
7.	$A = 28° 14.7'$,	$C = 109° 32.5'$,	$b = 730.80$.
8.	$B = 102° 38.3'$,	$C = 20° 3.2'$,	$b = 479.36$.
9.	$B = 30° 36.8'$,	$C = 107° 15.5'$,	$b = 0.14379$.
10.	$A = 36° 14.2'$,	$B = 14° 26.7'$,	$c = 16.583$.

11. One diagonal of a parallelogram is 21.871 inches. It makes angles of 43° 20.5′ and 56° 14.2′ respectively with the sides. Find the sides of the parallelogram.

12. At a certain point in the same horizontal plane as the base of a radio tower, the angle of elevation of the top of the tower is 13° 25.4′. At a point which is 156.25 feet nearer the tower the angle of elevation is 18° 10.5′. Find the height of the tower.

59. Logarithmic solution of Case II.

Case II can also be solved logarithmically by using the law of sines. The solution may be checked by formula (1) of section 52 (page 83) or by the law of tangents. (See section 60.)

Example.

Solve the triangle $A = 38° 14.2'$, $a = 8.7161$, $b = 9.7869$.

SOLUTION.	a	8.7161
$\sin B = \dfrac{b \sin A}{a}$,	b	9.7869
	A	38° 14.2′
$\log \sin B$	$\log\ b$	0.99065
$= \log b + \log \sin A$	$\log \sin A$	9.79163 − 10
$+ \operatorname{colog} a.$	$\operatorname{colog} a$	9.05968 − 10
	$\log \sin B$	9.84196 − 10
$C = 180° - (A + B).$	B	**44° 1.5′**, B' = **135° 58.5′**
	A	38° 14.2′ 38° 14.2′
	$A + B$	82° 15.7′ 174° 12.7′
	C	**97° 44.3′**, C' = **5° 47.3′**
$c = \dfrac{a \sin C}{\sin A}$,	$\log \sin C$	9.99602 − 10
	$\log\ a$	0.94032
$\log c$	$\operatorname{colog} \sin A$	0.20837
$= \log a + \log \sin C$	$\log \sin C'$	9.00369 − 10
$+ \operatorname{colog} \sin A.$	$\log c$	1.14471
	$\log c'$	0.15238
	c	**13.954**
	c'	**1.4203**
CHECK. 1st solution.	$b + a$	18.5030
$\dfrac{b+a}{c} = \dfrac{\cos \frac{1}{2}(B - A)}{\sin \frac{1}{2}C} = x,$	$B - A$	5° 47.3′
	$\frac{1}{2}(B - A)$	2° 53.65′
$\log x = \log(b + a) - \log c,$	$\frac{1}{2}C$	48° 52.15′
	$\log(b + a)$	1.26724
	$\log\ c$	1.14471
	$\log\ x$	0.12253
$\log x$	$\log \cos \frac{1}{2}(B - A)$	9.99944 − 10
$= \log \cos \frac{1}{2}(B - A)$	$\log \sin \frac{1}{2}C$	9.87692 − 10
$- \log \sin \frac{1}{2}C.$	$\log\ x$	0.12252

EXERCISES VII. F

Solve all possible triangles in the following set, and find their areas:

1. $a = 62.518$, $b = 72.932$, $B = 98° 23.5'$.
2. $a = 429.15$, $c = 328.12$, $A = 130° 33.7'$.
3. $b = 3912.7$, $c = 3526.5$, $C = 35° 25.8'$.
4. $b = 12968$, $c = 1529.6$, $B = 38° 28.6'$.
5. $a = 86.425$, $c = 73.463$, $C = 49° 18.9'$.
6. $b = 223.46$, $c = 327.92$, $C = 116° 19.6'$.

7.	$b = 0.32492,$	$c = 0.52392,$	$B = 27° 49.3'.$
8.	$a = 5660.1,$	$c = 8442.0,$	$A = 42° 6.2'.$
9.	$b = 45.872,$	$c = 56.321,$	$B = 20° 14.5'.$
10.	$a = 57.147,$	$b = 46.703,$	$B = 19° 17.8'.$
11.	$a = 515.55,$	$c = 524.31,$	$A = 80° 52.2'.$

12. Two lighthouses are 3.276 miles apart, and a certain rock is 4.835 miles from one of them. The angle subtended by the two lighthouses at the rock is 15° 22′. How far is the rock from the other lighthouse? (Two solutions.)

13. The diagonals of a parallelogram intersect at an angle of 52° 10.2′. One diagonal is 3325 feet and one side is 2995 feet. Find the other diagonal. (Two solutions.)

60. Law of tangents.

Case III was solved by the law of cosines, but the method is not adapted to the use of logarithms. In the present sec-

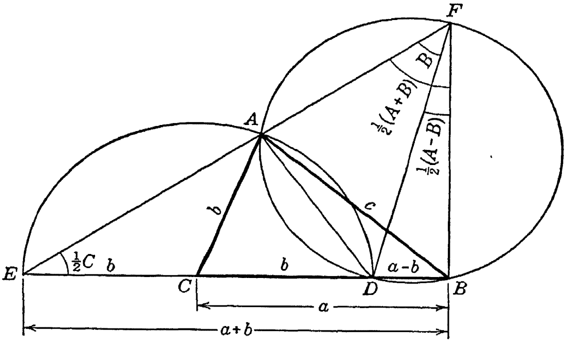


Fig. 52

tion we shall develop a formula which enables us to use logarithms in solving this case.

In triangle ABC, suppose that a is greater than b (Fig. 52). With C as center and b as radius, draw a circle cutting BC in D, and BC extended in E. Then,

$$BD = a - b, \qquad BE = a + \tag{1}$$

At B draw a perpendicular to BE. Draw EA and extend to meet this perpendicular in F. On DF as diameter construct a circle. This circle will pass through A; for FAD is a right angle, since it is supplementary to EAD, which is inscribed in a semicircle. The circle will also pass through B, since DBF is a right angle by construction.

It follows that $BEA = \frac{1}{2}C$, and that $BFE = \frac{1}{2}(A + B)$, since BFE is the complement of $\frac{1}{2}C$. Also, DFA and B are equal, since they are inscribed angles intercepting the same arc, AD. By subtraction we find $BFD = \frac{1}{2}(A - B)$.

Now in right triangles BDF and BEF we have respectively,

$$\frac{a-b}{BF} = \tan \tfrac{1}{2}(A - B), \qquad \frac{a+b}{BF} = \tan \tfrac{1}{2}(A + B). \tag{2}$$

Dividing the first of the foregoing equations by the second, we obtain

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}. \tag{3}$$

This formula is one form of the **law of tangents.** Other forms may be obtained by a cyclic change of letters. If b were greater than a, we could interchange a and b, A and B, in (3). If a and b were equal the formula would still hold, but would be trivial, since both sides of the equation would be zero.

★61. Mollweide's equations.

From Fig. 52 we can obtain two formulas which are very serviceable in checking solutions of triangles.

Applying the law of sines to triangle ABD, we get

$$\frac{a-b}{c} = \frac{\sin DAB}{\sin BDA} \tag{1}$$

But $DAB = \frac{1}{2}(A - B)$, since DAB and DFB are inscribed angles intercepting the same arc, BD; and BDA

$= 90° + \frac{1}{2}C$, since BDA is an exterior angle of the triangle ADE. Since $\sin(90° + \frac{1}{2}C) = \cos \frac{1}{2}C$, (1) reduces to

$$\frac{a - b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}. \tag{2}$$

Applying the law of sines to triangle ABE, we get

$$\frac{a + b}{c} = \frac{\sin BAE}{\sin \frac{1}{2}C}. \tag{3}$$

But $BAE = A + \frac{1}{2}C = \frac{1}{2}(A + B + C) + \frac{1}{2}(A - B) = 90° + \frac{1}{2}(A - B)$. Thus, $\sin BAE = \cos \frac{1}{2}(A - B)$, and (3) becomes

$$\frac{a + b}{c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}. \tag{4}$$

Formulas (2) and (4) are sometimes called **Mollweide's equations.*** Their advantage as checking formulas is that each contains all six parts of a triangle, and hence an error will be detected by a lack of agreement between the two members of one of these equations.

62. Logarithmic solution of Case III.

We are now ready to solve Case III by means of logarithms. The two angles are found by the law of tangents; the third side is then found by the law of sines. A check may be made by the law of sines or by one of Mollweide's equations.

Example.

Solve the triangle $a = 55.138$, $b = 33.094$, $C = 30° 24.6'$.

SOLUTION.

$$A + B = 180° - C.$$

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B),$$

* The law of tangents can be obtained from Mollweide's equations by division.

$$\log \tan \tfrac{1}{2}(A - B) = \log(a - b) + \operatorname{colog}(a + b) + \log \tan \tfrac{1}{2}(A + B).$$

a	55.138
b	33.094
C	30° 24.6′
$a - b$	22.044
$a + b$	88.232
$A + B$	149° 35.4′
$\frac{1}{2}(A + B)$	74° 47.7′
$\log(a - b)$	1.34329
$\operatorname{colog}(a + b)$	8.05437 − 10
$\log \tan \frac{1}{2}(A + B)$	0.56577
$\log \tan \frac{1}{2}(A - B)$	9.96343 − 10
$\frac{1}{2}(A - B)$	42° 35.4′
$+ B)$	74° 47.7′
A	**117° 23.1′**
B	**32° 12.3′**

$$c = \frac{b \sin C}{\sin B},$$

$$\log c = \log b + \log \sin C + \operatorname{colog} \sin B.$$

$\log b$	1.51975
$\log \sin C$	9.70431 − 10
$\operatorname{colog} \sin B$	0.27331
$\log c$	1.49737
c	**31.432**

CHECK.

$$c = \frac{a \sin C}{\sin A},$$

$$\log c = \log a + \log \sin C + \operatorname{colog} \sin A.$$

$\log a$	1.74145
$\log \sin C$	9.70431 − 10
$\operatorname{colog} \sin A$	0.05162
$\log c$	1.49738

EXERCISES VII. G

Solve the following triangles, and find their areas:

1. $a = 284.3,\quad b = 286.5,\quad C = 63° 38'.$

2. $a = 49.366,\quad b = 26.437,\quad C = 47° 16.6'.$

3. $a = 36.508$, $b = 8.9156$, $C = 132° 18.3'$.
4. $b = 247.81$, $c = 513.58$, $A = 147° 8.8'$.
5. $a = 67.375$, $c = 36.858$, $B = 12° 28.5'$.
6. $b = 284.12$, $c = 362.12$, $A = 126° 32.2'$.
7. $a = 482.33$, $c = 395.71$, $B = 137° 31.2'$.
8. $a = 0.06350$, $c = 0.10391$, $B = 83° 29.4'$.
9. $b = 17976$, $c = 24824$, $A = 43° 36.2'$.
10. $a = 4216.4$, $b = 3125.2$, $C = 88° 10.1'$.
11. Two points, A and B, are at opposite ends of a lake. To find the distance between them, a point C is selected so that it is possible to measure a straight line from A to C and also from B to C. The distances AC and BC are measured and found to be 3472 feet and 2956 feet respectively. The angle ACB is measured by means of a transit, and is found to be 46° 25′. What is the distance from A to B?
12. Two sides of a triangular plot of ground are 256.8 feet and 198.2 feet respectively, the included angle being 65° 22′. Find (a) the length of fence required to enclose the plot, (b) the area of the plot.

★63. Heron's formula.

In this section and the following we shall derive formulas for the logarithmic solution of Case IV.

From formula (7) of section 51 we have

$$(\text{area})^2 = \tfrac{1}{4}b^2c^2 \sin^2 A, \tag{1}$$

and, since by exercise I. C, 24,*

$$\sin^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A),$$

we have

$$(\text{area})^2 = \tfrac{1}{4}b^2c^2(1 + \cos A)(1 - \cos A). \tag{2}$$

By the law of cosines,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{3}$$

* This exercise covers only the case in which A is acute. The case in which A is obtuse is covered by (4) of section 68.

and consequently,

$$1 + \cos A = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}$$
$$= \frac{(b + c + a)(b + c - a)}{2bc}, \tag{4}$$

$$1 - \cos A = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$
$$= \frac{(a + b - c)(a - b + c)}{2bc}. \tag{5}$$

If we let

$$s = \tfrac{1}{2}(a + b + c), \tag{6}$$

then it can easily be shown that

$$b + c - a = 2(s - a), \qquad a + c - b = 2(s - b), \qquad a + b - c = 2(s - c). \tag{7}$$

Making use of (6) and (7) in (4) and (5), we find that

$$1 + \cos A = \frac{2s(s - a)}{bc}, \qquad 1 - \cos A = \frac{2(s - b)(s - c)}{bc} \tag{8}$$

Substituting these values in (2) and extracting the square root, we obtain **Heron's formula** for the area of a triangle:

$$\textbf{area} = \sqrt{s(s - a)(s - b)(s - c)}, \tag{9}$$

in which s is defined by (6), that is, it is the semiperimeter of the triangle.

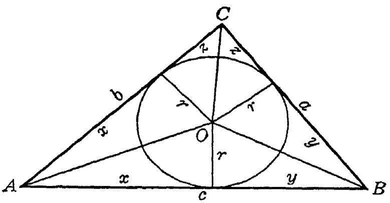


FIG. 53

64. Half-angle formulas.

In Fig. 53 the radius of the circle inscribed in triangle ABC is r. Then r is the altitude of each of the triangles AOB, BOC, COA, which have as a common vertex the center, O, of the circle. It

is readily seen that the area of the triangle ABC is given by the formula

$$\text{area} = \tfrac{1}{2}r(a + b + c) = rs, \tag{1}$$

where, as before $s = \frac{1}{2}(a + b + c)$.

But, by Heron's formula,

$$\text{area} = \sqrt{s(s - a)(s - b)(s - c)}. \tag{2}$$

Equating the two expressions for the area, we find that

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}. \tag{3}$$

Now let the equal tangents from A be denoted by x, those from B by y, and those from C by z. Adding all of these tangents, we get the perimeter of the triangle, or

$$2x + 2y + 2z = a + b + c = 2s. \tag{4}$$

From this it follows that $x + y + z = s$, and

$$x = s - y - z = s - a, \qquad y = s - b, \qquad z = s - c.$$

Consequently,

$$\tan \tfrac{1}{2}A = \frac{r}{s - a}, \quad \tan \tfrac{1}{2}B = \frac{r}{s - b}, \quad \tan \tfrac{1}{2}C = \frac{r}{s - c}, \tag{5}$$

in which r is given by (3), and

$$s = \tfrac{1}{2}(a + b + c). \tag{6}$$

Formulas (5) may be termed the **half-angle formulas.**

65. Logarithmic solution of Case IV.

The half-angle formulas enable us to use logarithms in solving Case IV.

Example.

Solve the triangle $a = 51.286$, $b = 65.353$, $c = 20.001$.

SOLUTION.

$$s = \tfrac{1}{2}(a + b + c).$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) + \operatorname{colog} s].$$

$$\tan \tfrac{1}{2}A = \frac{\quad}{s-a}, \text{ etc.,}$$

$$\log \tan \tfrac{1}{2}A = \log r - \log(s-a), \text{ etc.}$$

a	51.286
b	65.353
c	20.001
$2s$	136.640
s	68.320
$s-a$	17.034
$s-b$	2.967
$s-c$	48.319
CHECK. s	68.320
$\log(s-a)$	1.23131
$\log(s-b)$	0.47232
$\log(s-c)$	1.68412
$\operatorname{colog} s$	8.16545 − 10
$\log r^2$	1.55320
$\log r$	0.77660
$\log \tan \frac{1}{2}A$	9.54529 − 10
$\log \tan \frac{1}{2}B$	0.30428
$\log \tan \frac{1}{2}C$	9.09248 − 10
$\frac{1}{2}A$	19° 20.4′
$\frac{1}{2}B$	63° 36.4′
$\frac{1}{2}C$	7° 3.2′
A	**38° 40.8′**
B	**127° 12.8′**
C	**14° 6.4′**
$A + B + C$	

CHECK. $A + B + C = 180°$.

It is an easy and valuable check to add the values of $s - a$, $s - b$, and $s - c$, as soon as these have been found. Since this gives $3s - a - b - c = 3s - 2s = s$, the sum should be equal to s. This simple check often prevents working the entire problem with an incorrect value for one of the expressions $s - a$, $s - b$, $s - c$.

For convenience in computing $\log \tan \frac{1}{2}A$, etc., $\log r$ may be written at the bottom of a slip of paper, and placed in turn above $\log(s - a)$, $\log(s - b)$, $\log(s - c)$.

EXERCISES VII. H

Solve the following triangles, and find their areas:

1. $a = 125.36$, $b = 176.43$, $c = 101.23$.

2.	$a = 23.586$,	$b = 25.743$,	$c = 10.047$.
3.	$a = 10.057$,	$b = 19.436$,	$c = 15.067$.
4.	$a = 2249.8$,	$b = 2467.2$,	$c = 3152.6$.
5.	$a = 50014$,	$b = 70023$,	$c = 90054$.
6.	$a = 121.62$,	$b = 9.8210$,	$c = 113.94$.
7.	$a = 42.391$,	$b = 23.168$,	$c = 51.833$.
8.	$a = 0.98452$,	$b = 0.67514$,	$c = 0.81106$.
9.	$a = 1.8943$,	$b = 2.2465$,	$c = 3.5488$.
10.	$a = 0.11056$,	$b = 0.05264$,	$c = 0.17842$.

11. The sides of a triangular lot are 156.8 feet, 132.4 feet, and 148.3 feet respectively. Find the radius of the largest upright cylindrical tank that can be constructed on the lot.

12. In a triangle ABC, $a = 25.864$, $b = 26.232$, and the median from A is 20.866. Find the angles of the triangle, also side c.

66. Summary of methods.

The methods of solving oblique triangles are recapitulated below.

Case I. Two angles and a side given.	Use **law of sines.** Check by Mollweide's equation.
Case II. Two sides and the angle opposite one of them given.	Use **law of sines.** (Law of cosines may be used.) Note number of solutions. Check by Mollweide's equation.
Case III. Two sides and the included angle given.	If the sides are given to a small number of significant figures, or if only the third side is desired, **law of cosines** may be used. Find angles by **law of sines.** For logarithmic solution, use **law of tangents** to find angles. Find third side by **law of sines.** Check by Mollweide's equation.
Case IV. Three sides given.	If the sides are given to a small number of significant figures, or if only one angle is desired, **law of cosines** may be used. For logarithmic solution, use **half-angle formulas.** Check by $A + B + C = 180°$.

Note that an alternative check to Mollweide's equations is provided by the law of tangents.

To find the area of a triangle we can always resort to the fundamental formula of half the product of the base and the altitude. However, the formula

$$\text{area} = \tfrac{1}{2}bc \sin A$$

(and the others obtained from it by a cyclic change of letters) and Heron's formula are sometimes useful. (See also exercise VII. I, 47.)

MISCELLANEOUS EXERCISES VII. I

Solve the following triangles, and find their areas:

1.	$A = 55° 23.2'$,	$B = 72° 20.9'$,	$a = 537.14$.
2.	$A = 87° 58.4'$,	$a = 119.51$,	$b = 72.486$.
3.	$B = 19° 58.4'$,	$C = 94° 39.8'$,	$a = 4.3612$.
4.	$A = 34° 39.6'$,	$b = 61.519$,	$c = 47.612$.
5.	$a = 0.74261$,	$b = 0.10398$,	$c = 0.67517$.
6.	$C = 11° 14.3'$,	$b = 14.433$,	$c = 9.4670$.
7.	$C = 26° 36.6'$,	$a = 273.18$,	$b = 479.63$.
8.	$a = 1960.4$,	$b = 1093.3$,	$c = 2601.3$.
9.	$B = 127° 9.3'$,	$a = 67517$,	$c = 10398$.
10.	$B = 32° 18.0'$,	$a = 480.01$,	$b = 312.39$.
11.	$A = 53° 7.8'$,	$C = 45° 40.0'$,	$b = 374.85$.
12.	$B = 73° 44.4'$,	$C = 87° 20.1'$,	$c = 712.25$.
13.	$B = 104° 15.0'$,	$a = 7.3515$,	$c = 4.9764$.
14.	$B = 75° 45.0'$,	$a = 735.15$,	$b = 983.97$.
15.	$a = 31.628$,	$b = 68.235$,	$c = 52.063$.
16.	$a = 592.45$,	$b = 285.77$,	$c = 585.48$.
17.	$A = 43° 36.2'$,	$B = 102° 40.8'$,	$c = 392.37$.
18.	$C = 43° 35.6'$,	$b = 74.591$,	$c = 34.191$.
19.	$C = 51° 59.9'$,	$a = 228.15$,	$b = 109.84$.
20.	$a = 0.45562$,	$b = 0.32897$,	$c = 0.43129$.

21. Two sides of a parallelogram are 694.50 feet and 418.32 feet respectively; one diagonal is 602.94 feet. Find the length of the other diagonal.

22. The bases of a trapezoid are 397.62 and 254.15 respectively;

the angles that the sides make with the longer base are $68° 39.2'$ and $72° 6.0'$. Find the sides and the diagonals.

23. The sides of a triangular field are $AB = 193.8$ feet, $BC = 139.8$ feet, and $CA = 218.3$ feet. If the bearing of AB is N 20° E,* find the bearings of BC and CA, it being given that C is west of AB.

24. Let A, B, C represent three consecutive mileposts on a straight road. From each of these a distant spire is observed. At A it is northeast, at B it is east, and at C it is E 30° S. Find the distance of the spire from B, and the shortest distance from the road to the spire.

25. Along one bank of a river with parallel banks, a surveyor lays off a base line, AB, 600.0 feet long. From each end of the line an object C on the opposite bank is sighted. The angles which the lines of sight make with the base line are $62° 5.3'$ and $81° 34.7'$ respectively. Find the width of the river.

26. Points A and B are on opposite sides of a body of water, and soundings are to be taken in the line AB at points one-quarter, one-half, and three-quarters of the distance from A to B. On the shore, a base line AC is laid off, and it is found that angle $BAC = 63° 19'$, angle $ACB = 78° 43'$. What angles must be turned from CA at C in order to line up the boat from which the soundings are made at the proper points on the line AB?

27. In order to measure the distance between two inaccessible points, A and B, a base line, CD, 1168.2 feet in length was laid off. The following angles were then measured: $ACD = 132° 29'$, $ACB = 82° 20'$, $ADC = 45° 59'$, $BDC = 124° 48'$. Find the distance AB.

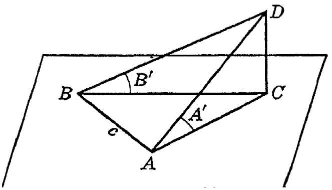


Fig. 54

28. It is required to find the horizontal distance and the vertical distance from a point A to an inaccessible point D, when it is not convenient to measure a base line in the same vertical plane with D. (See Fig. 54.) Draw AB, of length c, in any

* This means that the line drawn from A to B makes an angle of 20° with north, measured toward east.

convenient direction, in a horizontal plane. Let C be the foot of the perpendicular from D to this plane. Let A' and B' be the angles of elevation of D from A and B respectively. Show that

$$AC = \frac{c \sin B}{\sin C}, \qquad BC = \frac{c \sin A}{\sin C},$$

$$CD = \frac{c \sin A \tan B'}{\sin C} = \frac{c \sin B \tan A'}{\sin C}$$

where A, B, C are the angles of the triangle ABC. The height CD can be found from both formulas in order to check.

29. In the preceding exercise let $AB = 1255$ feet, $ABC = 46° 27'$, $BAC = 54° 40'$, $A' = 38° 42'$. Find AC, CD, B'.

30. Two boundary lines of a piece of property intersect at an angle of 85°. It is desired to cut off a triangular portion of the property which will be one acre (43560 square feet) in area by means of a straight fence. If the fence begins at a point on one boundary 250 feet from the corner of the property, and runs in a straight line to the other boundary, what angles does it make with the boundary lines, and how long is it?

31. To measure across a pond from A to B, a point C is selected so that $AC = 489$ feet, $BC = 674$ feet, and angle $ACB = 78° 45'$. Find the distance AB.

32. The diagonals of a parallelogram are 56.5 yards and 78.4 yards respectively. They intersect at an angle of 51° 35′. Find the area of the parallelogram.

33. A chimney projects 6 feet above a roof. At a point 10 feet 8 inches down the roof from the base of the chimney, the chimney subtends an angle of 17° 40′. Find the angle at which the roof is inclined to the horizontal.

34. The sides of a triangle are 14.832, 16.987, 18.645 respectively. Find the length of the perpendicular from the vertex of the largest angle to the side opposite.

35. The sides of a triangular grass plot are 47.5, 64.5, and 85 feet respectively. Find the minimum radius of action of an automatic lawn sprinkler which will water all parts of the plot simultaneously.

36. Find the radius of the largest circular flower bed which can be constructed on the plot of the preceding exercise.

37. The sum of the sides of a triangle is 100 inches. The angles are in the continued proportion 1 : 2 : 4. Find the sides.

38. Find the number of square yards of canvas in a conical tent, if the angle between the axis of the cone and an element is 30°, and the center pole is 14 feet high.

39. The sides of a triangular field which contains 15 acres are in the continued proportion 3 : 5 : 7. Find the sides. (1 acre = 160 sq. rd.)

40. Prove that the area of a quadrilateral is equal to half the product of its diagonals multiplied by the sine of their included angle.

41. A point A is in the same horizontal plane as the base of a radio tower. From this point a horizontal line AB, of length d, is drawn directly toward the tower. If the angle of elevation of the top of the tower from the point A is denoted by A, and the angle of elevation from the point B is denoted by B, show that the height of the tower is

$$\frac{d \sin A \sin B}{\sin(B - A)}.$$

42. A flagpole of height k stands on top of a building. From a certain point of observation in the same horizontal plane as the base of the building, the angle of elevation of the top of the pole is A, the angle of elevation of the bottom of the pole is B. Show that the distance d to the building from the point of observation, and the height h of the building are

$$d = \frac{k \cos A \cos B}{\sin(A - B)}; \qquad h = \frac{k \cos A \sin B}{\sin(A - B)}.$$

43. In a triangle ABC, D is the intersection of the median from A and the bisector of angle C. Prove that

$$a \times \text{area } ABC = (a + 2b) \times \text{area } BCD.$$

44. On the sides of a triangle ABC are constructed isosceles triangles with their vertices on the circumference of the circumscribed circle of the given triangle. Show that their areas are in the ratio

$$\frac{a^2}{s-a} = \frac{b^2}{s-b} : \frac{c^2}{-c}$$

where $s = \frac{1}{2}(a + b + c)$.

45. Prove the formulas:

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \cos \tfrac{1}{2}A - \sqrt{\frac{s(s-a)}{bc}}$$

46. Prove that the area of a triangle is given by the formula

$$\frac{c^2 \sin A \sin B}{2 \sin(A + B)}.$$

47. Prove that the area of a triangle is given by the formula $abc/4R$, where R is the radius of the circumscribed circle.

48. Find the angle between the diagonal of a cube and the diagonal of a face of the cube, both diagonals drawn from the same vertex.

49. From one corner of a cube lines are drawn in two of its faces, making angles of 30° and 40° respectively with the common edge of these faces. Find the angle between the two lines.

50. A rectangular solid is 5 inches long, 4 inches wide, and 3 inches high. From one vertex a diagonal is drawn in each of the three faces having this vertex in common. Find the angles between these diagonals.

★67. Vectors.

If an object is at the point A in Fig. 55, and is displaced (i.e. moved) to the point B, the displacement may be represented by the directed line segment AB. (The arrow indicates the direction.) It will be noted that this line segment represents both the amount and the direction of the displacement. Now let BC represent another displacement. If an object originally at A is given both of these displacements it will arrive at the point C. The order in which these displacements occur is immaterial; that is, the object may be moved from A to B and then from B to C, or it may be

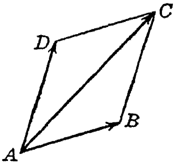


FIG. 55

moved from A to D (the displacement AD is equal and parallel to BC) and then from D to C. The displacement AC is called the **resultant** of the displacements AB and AD. (Cf. section 9.) Obviously, the resultant is a diagonal of the parallelogram of which AB and AD are sides. The displacements AB and AD are called **components** of AC.

It can be proved experimentally that two forces acting at the same point also combine into a resultant according to this so-called parallelogram law. Thus, if in Fig. 55, AB and AD represent, in magnitude and direction, two forces acting on an object at A, then the diagonal AC will represent, in magnitude and direction, the resultant of the two given forces. That is, the single force represented by AC will have the same effect on the object as the two forces represented by AB and AD.

Velocities and many other directed quantities (those which have direction as well as magnitude) also combine according to the parallelogram law. Such a quantity is called a **vector quantity.** The directed line segment representing the vector quantity is called a **vector.**

The resultant of any two vectors may of course be found graphically or geometrically by completing the parallelogram of which they form the adjacent sides, and drawing the diagonal. This is called the " addition " of the vectors. They may also be " added " by placing the initial point of one on the terminal point of the other, preserving the proper direction of each, and then drawing a third vector from the initial point of the first to the terminal point of the second. This can be seen by reference to Fig. 55.

A knowledge of trigonometry is essential in dealing with vectors. Its application may be illustrated by the following examples.

Example 1.

Three forces of 20, 30, and 40 pounds, respectively, are in equilibrium. Find the angles that they make with each other.

Solution. Since the forces are in equilibrium, any one of them must be equal in magnitude and opposite in direction to the resultant of the other two. That is, we have a parallelogram in which the diagonal is, for example, 40, and in which the two sides are 20 and 30. (See Fig. 56.) Our problem is thus reduced to that of finding the angles of a triangle whose sides are 20, 30, and 40. This may be done by employing the law of cosines or the law of tangents. Since the numbers are simple, we shall use the former. Referring to the figure, we see that

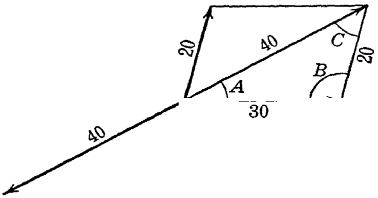


Fig. 56

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(40)^2 + (30)^2 - (20)^2}{2 \cdot 40 \cdot 30} = 0.8750,$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{(30)^2 + (20)^2 - (40)^2}{2 \cdot 30 \cdot 20} = -0.2500,$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(20)^2 + (40)^2 - (30)^2}{2 \cdot 20 \cdot 40} = 0.6875;$$

$$A = 28^\circ\, 57', \qquad B = 104^\circ\, 29', \qquad C = 46^\circ\, 34'.$$

Check. $\qquad A + B + C = 180^\circ\, 00'.$

Therefore,

$$\begin{aligned}
&\text{angle between 40-lb. and 30-lb. forces} = 180^\circ - A = 151^\circ\ \ 3',\\
&\text{angle between 30-lb. and 20-lb. forces} = 180^\circ - B = \ \ 75^\circ\, 31',\\
&\text{angle between 20-lb. and 40-lb. forces} = 180^\circ - C = \underline{133^\circ\, 26'}.\\
&\hspace{20em}\text{Check.}\quad 360^\circ\, 00'.
\end{aligned}$$

It may be noted that since the forces are represented by the sides of the triangle ABC, the forces are proportional to the sines of the opposite angles.

Example 2.

An airplane having a speed of 120 miles an hour in calm air is pointed in a direction 30° east of north. A wind having a velocity

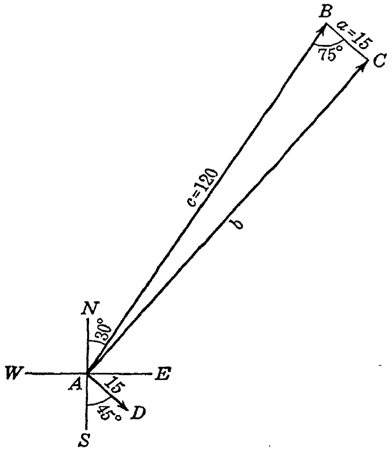


FIG. 57

of 15 miles an hour is blowing from the northwest. Find the speed and direction of the airplane relative to the ground.

SOLUTION. Referring to Fig. 57, we see that the vector AB represents the velocity of the airplane due to its own power, and that the vector AD represents the velocity of the wind. We draw BC parallel and equal to AD, and connect A and C. Then AC represents the velocity of the airplane relative to the ground and is the vector required.

It is readily seen, if we draw a north-south line through B, that angle $B = 30° + 45° = 75^{c}$. Thus, in the triangle ABC, we have $a = 15$, $c = 120$, $B = 75°$. The numbers are simple, and we use the law of cosines, finding

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= (15)^2 + (120)^2 - 2 \cdot 15 \cdot 120 \cdot \cos 75° \\ &= 13693.25, \\ b &= 117.0. \end{aligned}$$

Further,

$$\sin A = a \sin B \quad \frac{15 \sin 75°}{117.0} = 0.1238,$$

$$A\ (= BAC) = 7°\,7', \qquad NAC = 30° + 7°\,7' = 37°\,7'.$$

Thus, the airplane actually travels in a direction 37° 7′ east of north at a speed of 117 miles per hour relative to the ground.

EXERCISES VII. J

1. Two forces of 8 and 11 pounds respectively act at an angle of 75° with each other. Find the magnitude of their resultant, and the angle that it makes with the 8-pound force.
2. Three forces of 7, 9, and 13 pounds respectively are in equilibrium. Find the angles that they make with each other.

3. A train is traveling at the rate of 30 miles an hour, and rain is falling with a velocity of 22 feet a second, at an angle of 30° with the vertical and in the same direction as the motion of the train. Find the direction of the splashes made on the windows of the coaches by the raindrops.
4. A motorboat which has a speed of 15 miles an hour in still water sets out to cross a stream which has a current of 5 miles an hour. The boat points upstream at an angle of 30° with the bank. Find its actual speed and the actual direction that it takes.
5. If a force of 100 pounds is resolved into components of 60 pounds and 50 pounds respectively, what angle do these components make with each other?
6. An airplane has a speed of 150 miles an hour in still air. The pilot wishes to fly in a direction 65° east of north. A 15-mile wind is blowing from the southeast. In what direction must the airplane be pointed?
7. The actual velocity of a motorboat is 25 miles an hour due north. The wind is blowing from the direction N 50° W at the rate of 15 miles an hour. What is the apparent velocity of the wind, and from what direction does it seem to strike the boat?
8. Two forces of 475 and 530 pounds respectively, making an angle of 36° 35′ with each other, act at the same point. Find the magnitude of their resultant, and the angle that it makes with the smaller force.
9. Three forces of 255, 320, and 195 pounds respectively are in equilibrium. What angles do they make with each other?
10. An airplane has a speed of 120 miles an hour in still air. A 20-mile wind is blowing from the northwest. A pilot wishes to fly 200 miles west and return to his original position. In what direction must he point the airplane (a) on the outward trip? (b) on the return trip?

CHAPTER VIII

Trigonometric Formulas and Identities

68. Fundamental relations among the functions.

It is readily seen, from the generalized definitions of section 37, that the functions of any angle satisfy the same **reciprocal relations** as the functions of an acute angle, namely,

$$
\begin{aligned}
\csc\theta &= \frac{1}{\sin\theta}, & \sin\theta &= \frac{1}{\csc\theta},\\
\sec\theta &= \frac{1}{\cos\theta}, & \cos\theta &= \frac{1}{\sec\theta}, \qquad (1)\\
\cot\theta &= \frac{1}{\tan\theta}, & \tan\theta &= \frac{1}{\cot\theta}.
\end{aligned}
$$

The following relations can also be readily proved:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}. \qquad (2)$$

The first can be proved by making use of the definitions of the functions. For,

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta.$$

The second follows from the fact that $\cot\theta = 1/\tan\theta$, or it can be proved independently.

Starting from the equation

$$x^2 + y^2 = r^2, \qquad (3)$$

which may be obtained from Fig. 34 (page 67) by applying the theorem of Pythagoras, we can derive three more fundamental relations.

Dividing (3) by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

which, since $x/r = \cos\theta$ and $y/r = \sin\theta$, can be written

$$\mathbf{\cos^2\theta + \sin^2\theta = 1.} \tag{4}$$

Dividing (3) by x^2, we get

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2},$$

which becomes

$$\mathbf{1 + \tan^2\theta = \sec^2\theta.} \tag{5}$$

Finally, dividing (3) by y^2, we get

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2};$$

or

$$\mathbf{\cot^2\theta + 1 = \csc^2\theta.} \tag{6}$$

Relations (4), (5), (6) may be termed the **Pythagorean relations.** They may be written in different forms if desirable; for example, (4) may be transformed as follows:

$$\cos^2\theta = 1 - \sin^2\theta, \qquad \text{or} \qquad \cos\theta = \pm\sqrt{1 - \sin^2\theta}.$$

69. *Finding the other functions of an angle when one function is given.*

The foregoing formulas may be used to find the values of the functions of an angle when the value of one function is given. However, the method used in section 4 for functions of acute angles is preferable.

Example 1.

Given $\sin \theta = \frac{3}{5}$; find the other functions of θ.

Solution. Since $\sin \theta = y/r$, we may take $r = 5$, from which it follows that $y = 3$. Draw a circle with its center at the origin and having a radius of 5 units (Fig. 58). Take a point on the y-axis at a distance of 3 units above the x-axis. A line through this point parallel to the x-axis will cut the circle in two points, and consequently there will be two positions for the angle θ: θ_1 in quadrant I, and θ_2 in quadrant II, as shown in the figure.

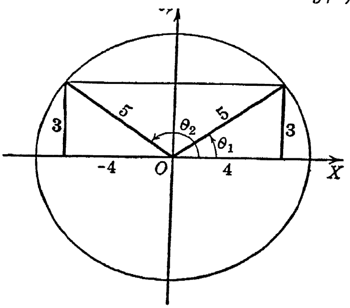


Fig. 58

Now,

$$x^2 = 5^2 - 3^2 = 16, \qquad x = \pm 4.$$

Thus, corresponding to the angle in quadrant I we have an abscissa 4, and corresponding to the angle in quadrant II we have an abscissa -4. We can now read all of the functions of both angles directly from the figure.

Quadrant I	Quadrant II
$\sin \theta_1 = \frac{3}{5}$,	$\sin \theta_2 = \frac{3}{5}$,
$\cos \theta_1 = \frac{4}{5}$,	$\cos \theta_2 = -\frac{4}{5}$,
$\tan \theta_1 = \frac{3}{4}$,	$\tan \theta_2 = -\frac{3}{4}$,
$\csc \theta_1 = \frac{5}{3}$,	$\csc \theta_2 = \frac{5}{3}$,
$\sec \theta_1 = \frac{5}{4}$,	$\sec \theta_2 = -\frac{5}{4}$,
$\cot \theta_1 = \frac{4}{3}$.	$\cot \theta_2 = -\frac{4}{3}$.

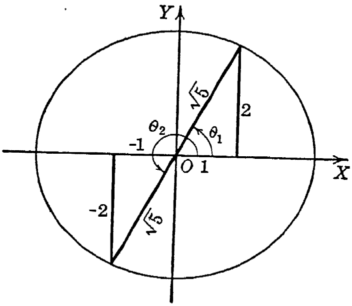


Fig. 59

Example 2.

Given $\tan \theta = 2$; find the other functions.

Solution. Since $\tan \theta = y/x$, we may take $y = 2$ and $x = 1$, or $y = -2$ and $x = -1$ (Fig. 59). There are two angles, one in quadrant I, the other in quadrant III. In either case,

$$r^2 = 1^2 + 2^2 = 5, \quad r = \sqrt{5}.$$

(We take only the positive square root as the value of r, according to the agreement of section 35.) From the figure we read

Quadrant I	**Quadrant III**
$\sin\theta_1 = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$,	$\sin\theta_2 = \dfrac{-2}{\sqrt{5}} = -\dfrac{2\sqrt{5}}{5}$,
$\cos\theta_1 = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$,	$\cos\theta_2 = \dfrac{-1}{\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$,
$\tan\theta_1 = 2$,	$\tan\theta_2 = \dfrac{-2}{-1} = 2$,
$\csc\theta_1 = \dfrac{\sqrt{5}}{2}$,	$\csc\theta_2 = \dfrac{\sqrt{5}}{-2} = -\dfrac{\sqrt{5}}{2}$,
$\sec\theta_1 = \sqrt{5}$,	$\sec\theta_2 = \dfrac{\sqrt{5}}{-1} = -\sqrt{5}$,
$\cot\theta_1 = \dfrac{1}{2}$.	$\cot\theta_2 = \dfrac{-1}{-2} = \dfrac{1}{2}$.

EXERCISES VIII. A

Find the other functions of θ, given that

1. $\sin\theta = \frac{12}{13}$, θ in quadrant I.
2. $\cos\theta = -\frac{4}{5}$, θ in quadrant III.
3. $\tan\theta = -\frac{2}{3}$, θ in quadrant IV.
4. $\cot\theta = \frac{1}{5}$, θ in quadrant III.
5. $\cos\theta = -\frac{2}{5}$, θ in quadrant II.
6. $\csc\theta = -\frac{41}{9}$, θ in quadrant IV.
7. $\sec\theta = \sqrt{2}$, θ in quadrant IV.
8. $\sin\theta = \frac{5}{6}$, θ in quadrant II.
9. $\tan\theta = \frac{7}{24}$, θ in quadrant III.
10. $\csc\theta = \frac{17}{15}$, θ in quadrant II.

Find the other functions of θ if

11. $\sin\theta = \frac{1}{2}$.
12. $\cos\theta = \frac{2}{3}$.
13. $\tan\theta = -\frac{2}{5}$.
14. $\csc\theta = \frac{4}{3}$.
15. $\cot\theta = \frac{5}{2}$.
16. $\sec\theta = \frac{5}{4}$.
17. $\sec\theta = -2$.
18. $\cos\theta = -\frac{1}{4}$.
19. $\tan\theta = 0.5$.
20. $\sin\theta = -0.8$.
21. $\csc\theta = 3$.
22. $\cos\theta = 0.2$.

23. $\tan\theta = -\sqrt{3}$.

24. $\csc\theta = -\frac{5}{3}$.

25. $\cos\theta = -\frac{1}{3}$.

26. $\tan\theta = -5$.

27. $\cot\theta = 0.1$.

28. $\sin\theta = -\frac{5}{6}$.

29. $\tan\theta = \sqrt{2}$.

30. $\cot\theta = 1$.

31. If $\sin\theta = \frac{7}{25}$ and $\cos\phi = \frac{15}{17}$, find all possible values of

(a) $\tan\theta + \tan\phi$, (b) $\cos\theta + \sin\phi$,

(c) $5\sin\theta - 2\sin\phi$, (d) $\sec\theta\tan\phi$,

(e) $\dfrac{1+\cot\theta}{\sin\phi}$. (f) $\dfrac{1-\cos}{1+\tan\phi}$,

(g) $(2+\cos\theta)(3-2\sin\phi)$, (h) $(m+n\tan\theta)(m+n\cot\phi)$.

32. If $\tan\theta = \frac{21}{20}$ and $\cot\phi = -\frac{9}{40}$, find all possible values of

(a) $\sin\theta + \sin\phi$, (b) $\cos\theta + \tan\phi$,

(c) $\frac{1}{3}\sin\theta + \frac{1}{5}\sin\phi$, (d) $\sec\theta(2 - 3\cos\phi)$,

(e) $\csc\theta\sec\phi$, (f) $\sin\theta\cos\phi + \cos\theta\sin\phi$,

(g) $\dfrac{\sec\phi}{1+\frac{1}{2}\cos\theta}$, (h) $\dfrac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

70. Identities.

Formulas (1), (2), (4), (5), (6) of section 68 are **identities,** in the sense that they are satisfied by all possible values of θ for which their left-hand and right-hand members are defined. By means of them it is possible to prove other identities, and consequently to change an expression involving trigonometric functions into a different but equivalent form which is more suitable for the purpose at hand.

Example 1.

Prove: $\tan\theta + \cot\theta = \sec\theta\csc\theta.$

Solution. To reduce the expression on the left to that on the right we first make use of (2) of section 68:

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}.$$

But by (6) of section 68, the last numerator is equal to 1, and the above expression reduces to

$$\frac{1}{\cos\theta\sin\theta},$$

which, because of the reciprocal relations, is equal to $\sec\theta\csc\theta$. Thus, we have reduced the left-hand side to the right-hand side and have consequently proved the identity.

Example 2.

Prove: $$\frac{1+\tan^2\theta}{\csc\theta} = \sec\theta\tan\theta.$$

SOLUTION. Applying the Pythagorean relation (5) of section 68 to the numerator on the left, we reduce the fraction to

$$\frac{\sec^2\theta}{\csc\theta} = \frac{\sec\theta\,\dfrac{1}{\cos\theta}}{\dfrac{1}{\sin\theta}} = \sec\theta\cdot\frac{\sin\theta}{\cos\theta}$$

This, by the first of equations (2) of section 68, reduces to $\sec\theta\tan\theta$, and the identity is established.

Ordinarily, in proving an identity, one must transform one side into the other. No general method of proof can be given. However, a thorough familiarity with the fundamental identities is essential. These should be kept constantly in mind, and careful consideration should be given to the question of which one of them is appropriate to the situation. There should also be kept in mind the expression toward which one is working. It is usually better to work with the more complicated side of the identity, endeavoring to reduce it to the form of the simpler side.

Frequently, if all functions are expressed in terms of sines and cosines, a clue will be obtained as to the next step to take.

If one side of the identity involves but one function, it may be best to express everything on the other side in terms of that function.

It is usually best to avoid radical expressions when possible.

EXERCISES VIII. B

Prove the following identities:

1. $\cos\theta\tan\theta = \sin\theta.$
2. $\cot\theta\cos\theta = \csc\theta - \sin\theta.$
3. $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}.$
4. $(\tan\theta - \sin\theta)^2 + (1-\cos\theta)^2 = (1-\sec\theta)^2.$
5. $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta.$
6. $\cot\theta + \tan\theta = \dfrac{\csc^2\theta + \sec^2\theta}{\csc\theta\sec\theta}$
7. $\dfrac{\sin\theta + \tan\theta}{\cot\theta + \csc\theta} = \sin\theta\tan\theta.$
8. $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta - \cot\theta.$
9. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2.$
10. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta.$
11. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta.$
12. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta.$
13. $\dfrac{\csc\theta}{\csc\theta - 1} + \dfrac{\csc\theta}{\csc\theta + 1} = 2\sec^2\theta.$
14. $\dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$
15. $\dfrac{\tan^2\theta}{\sec^2\theta} + \dfrac{\cot^2\theta}{\csc^2\theta} = 1.$
16. $\dfrac{\sin\theta + \cos\phi}{\sin\theta - \cos\phi} = \dfrac{\sec\phi + \csc\theta}{\sec\phi - \csc\theta}$
17. $(\tan\theta + \cot\phi)(\cot\theta - \tan\phi) = \cot\theta\cot\phi - \tan\theta\tan\phi.$
18. $(\tan\theta - \sec\phi)(\cot\theta + \cos\phi) = \tan\theta\cos\phi - \cot\theta\sec\phi.$
19. $\sin^2\theta(1+\cot^2\theta) = 1.$
20. $\cos\theta(1+\tan^2\theta) = \sec\theta.$
21. $\sin\theta(1+\cot^2\theta) = \csc\theta.$
22. $\dfrac{1+\sec\theta}{1-\sec\theta} = \dfrac{\cos\theta + 1}{\cos\theta - 1}.$
23. $\sec\theta - \sin\theta\tan\theta = \cos\theta.$

24. $\dfrac{1 - \tan^2}{1 - \cot^2 \theta} = 1 - \sec^2$

25. $\tan \theta + \tan(90° - \theta) = \sec \theta \csc \theta.$

26. $\dfrac{\tan \theta + \sin \theta}{\tan \theta - \sin \theta} \quad \dfrac{\sec \theta + 1}{\sec \theta - 1}$

27. $\dfrac{\sin \theta}{1 + \cos \theta} = \csc \theta - \cot \theta.$

28. $\sec^4 \theta - \tan^4 \theta \quad 1 + 2 \tan^2 \theta.$

29. $\dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos^2 \theta - \sin^2 \theta.$

30. $\dfrac{\tan \theta - \tan \phi}{\cot \theta - \cot \phi} \quad -\tan \theta \tan \phi.$

31. $\dfrac{\cos \theta}{\cos \theta \quad \sin \theta} - \dfrac{1}{1 - \tan}$

32. $\dfrac{\tan \theta}{\sin^2 \theta} = \pm\sqrt{\dfrac{1 +}{1 -} \dfrac{\tan^2 \theta}{\cos^2 \theta}}.$

33. $\dfrac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi.$

34. $(1 - \cos^2 \theta)(1 + \cot^2 \theta) = 1.$

35. $\dfrac{1}{\sec \theta + \tan \theta} = \sec \theta - \tan \theta.$

36. $\dfrac{\sin \theta + \tan \theta}{1 + \sec \theta} = \sin$

37. $\dfrac{\cos \theta}{\sec \theta} \quad \dfrac{\sin \theta}{\cot \theta} \quad \dfrac{\cos \theta \cot \theta - \tan \theta}{\csc \theta}$

38. $\dfrac{1}{1 - \sin \theta} - \dfrac{\cot \theta}{\cot \theta - \cos \theta}$

39. $\dfrac{\cos \theta}{1 - \tan \theta} + \dfrac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta.$

40. Express $\sin \theta$ in terms of $\tan \theta$.

Solution.

$$\frac{\sin \theta}{\cos \theta} = \tan \theta,$$

$$\frac{\sin \theta}{\pm\sqrt{1 - \sin^2 \theta}} \quad \tan \theta,$$

$$\frac{\sin^2 \theta}{1 - \sin^2} = \tan^2 \theta,$$

$$\sin^2 \theta = \tan^2 \theta - \tan^2 \theta \sin^2 \theta,$$

$$(1 + \tan^2 \theta) \sin^2 \theta = \tan^2 \theta,$$

$$\sin^2 \theta = \frac{\tan^2 \theta}{1 + \tan^2 \theta};$$

$$\sin \theta = \pm \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$$

The exercise can also be solved as follows: Draw a right triangle having an acute angle θ. Mark the opposite side $\tan \theta$, the adjacent side 1. Then the hypotenuse will be $\sqrt{1 + \tan^2 \theta}$. The value of $\sin \theta$ can now be read from the figure. (Cf. section 69.) The double sign should be used with the radical.

41. Construct a table giving each of the functions in terms of the other functions.

71. Directed line segments.

In defining rectangular coordinates, we introduced the idea of a positive and a negative direction on a line. Thus, the positive direction on the x-axis is to the right, the positive direction on the y-axis is upward. Any line, such as one of these axes, on which the positive direction has been specified, is a **directed line.** A portion of a directed line, such as AB in Fig. 60, is called a **directed line segment.** The point A may be called the **initial point** and the point B the **terminal point** of the line segment AB.

Fig. 60

Two line segments may be added by placing the initial point of the second on the terminal point of the first; the sum is the segment from the initial point of the first segment to the terminal point of the second. (It is immaterial which segment is considered the first and which the second.) The proper direction must, of course, be preserved for each segment.

Fig. 61

Thus, if A, B, C are points arranged in any order on a directed line, we may write

$$AB + BC = AC,$$

which merely states that if we go from A to B and then from B to C, we reach the same position that we reach by going directly from A to C.

Subtraction of two directed line segments is accomplished by changing the direction of the segment to be subtracted, and then proceeding as in addition.

Several segments can be added by carrying out successively the process described for two segments.

72. Functions of the sum and the difference of two angles.

To derive a formula for $\cos(\theta + \phi)$, place the angles θ and ϕ with reference to the coordinate axes as shown in Fig. 62. Take a point P on the terminal side of the angle $\theta + \phi$, and drop a perpendicular PQ to the terminal side of θ. Draw PM and QN perpendicular to the x-axis.

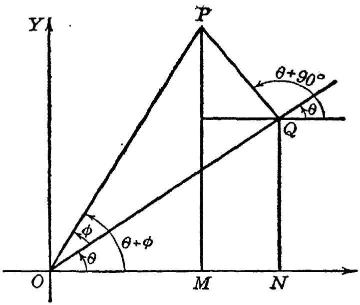


FIG. 62

Now, if we take into consideration the signs of the line segments involved, we have

$$OM = ON + NM. \tag{1}$$

But
$$\begin{aligned} OM &= OP\cos(\theta + \phi), \qquad ON = OQ\cos\theta, \\ NM &= QP\cos(90^\circ + \theta) = -QP\sin\theta. \end{aligned} \tag{2}$$

Substituting these values in (1), we get

$$OP\cos(\theta + \phi) = OQ\cos\theta - QP\sin\theta.$$

Division by OP gives

$$\cos(\theta + \phi) = \frac{OQ}{OP}\cos\theta - \frac{QP}{OP}\sin\theta.$$

But $$\frac{OQ}{OP} = \cos\phi, \qquad \frac{QP}{OP} = \sin\phi,$$

and consequently,

$$\boldsymbol{\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.} \tag{3}$$

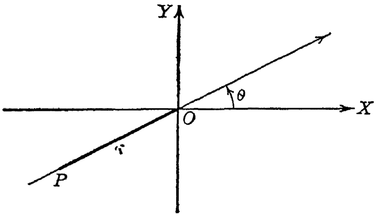


FIG. 63

The foregoing proof will hold for all values of θ and ϕ if we are careful to take into consideration the proper sign of each function and of each line segment involved. It will be necessary, however, to consider as negative a segment measured backward along the terminal side of an angle, such as segment OP in Fig. 63. In this figure r would be considered negative.

If in (3) we replace ϕ by $-\phi$, we get

$$\cos(\theta - \phi) = \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi),$$

or $$\boldsymbol{\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.} \tag{4}$$

To develop a formula for $\sin(\theta + \phi)$, we use (3), replacing θ by $90° - \theta$, and ϕ by $-\phi$. We get

$$\begin{aligned}\cos(90° - \theta - \phi) &= \cos[(90° - \theta) + (-\phi)]\\ &= \cos(90° - \theta)\cos(-\phi) - \sin(90° - \theta)\sin(-\phi),\end{aligned}$$

which becomes

$$\boldsymbol{\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.} \tag{5}$$

The foregoing formula can also be derived by dropping

perpendiculars from the points P and Q in Fig. 62 to the y-axis, and proceeding somewhat as in the proof of (3).

If in (5) we replace ϕ by $-\phi$, we get

$$\sin(\theta - \phi) = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi),$$

or
$$\boldsymbol{\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.} \tag{6}$$

Formulas (3) and (5) are sometimes called the **addition formulas** for the cosine and sine respectively. Similarly, (4) and (6) may be called their **subtraction formulas.**

To find the tangent of $\theta + \phi$ and of $\theta - \phi$, we proceed as follows:

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}.$$

If it is desired to express $\tan(\theta + \phi)$ in terms of $\tan\theta$ and $\tan\phi$, we divide numerator and denominator of the last fraction by $\cos\theta\cos\phi$, obtaining

$$\tan(\theta + \phi) \quad \frac{\dfrac{\sin\theta\cos\phi}{\cos\theta\cos\phi} + \dfrac{\cos\theta\sin\phi}{\cos\theta\cos\phi}}{\dfrac{\cos\theta\cos\phi}{\cos\theta\cos\phi} - \dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}}$$

which reduces to

$$\boldsymbol{\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.} \tag{7}$$

In like manner, or by replacing ϕ by $-\phi$ in (7), we find that

$$\boldsymbol{\tan(\theta - \phi) \quad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}} \tag{8}$$

For the cotangent we obtain the following formulas:

$$\cot(\theta + \phi) = \frac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta}, \tag{9}$$

$$\cot(\theta - \phi) = \frac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}. \tag{10}$$

Proofs of (9) and (10) are left as exercises.

EXERCISES VIII. C

1. Find $\sin 75°$ by setting $\theta = 45°$, $\phi = 30°$ in (5) of section **72**.

SOLUTION.
$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).\end{aligned}$$

2. Find $\cos 75°$, $\tan 75°$, $\cot 75°$.

3. Find $\sin 15°$, $\cos 15°$, $\tan 15°$, $\cot 15°$.

4. Verify the values of $\sin 90°$, $\cos 90°$, $\cot 90°$ by setting $\theta = 60°$, $\phi = 30°$ in (5), (3), (7), respectively, of section 72.

5. Verify the values of $\sin 30°$, $\cos 30°$, $\tan 30°$, $\cot 30°$ by setting $\theta = 60°$, $\phi = 30°$ in (6), (4), (8), (10), respectively, of section 72.

6. Find $\sin 105°$, $\cos 105°$, $\tan 105°$, $\cot 105°$.

7. Prove the formulas for $\sin(90° + \theta)$, $\cos(90° + \theta)$, $\tan(90° + \theta)$, $\cot(90° + \theta)$ by means of the addition formulas.

8. Prove the formulas for $\sin(180° - \theta)$, $\cos(180° - \theta)$, $\tan(180° - \theta)$, $\cot(180° - \theta)$ by means of the subtraction formulas.

Simplify the following expressions:

9. $\sin(\theta + 30°) + \cos(\theta + 60°)$.

10. $\sin(\theta + 60°) - \cos(\theta + 30°)$.

11. $\tan(\theta + 45°) + \cot(\theta - 45°)$.

12. $\cos(30° - \theta) - \cos(30° + \theta)$.

Prove the following identities:

13. $\sin(\theta + \phi) \sin(\theta - \phi) = \sin^2 \theta - \sin^2 \phi$.

14. $\cos(\theta + \phi) \cos(\theta - \phi) = \cos^2 \theta - \sin^2 \phi$.

15. $\tan(45° + \theta) = \dfrac{1 + \tan \theta}{1 - \tan \theta}$.

16. $\sin(45° + \theta) \cos(45° + \theta) = \frac{1}{2}(\cos^2 \theta - \sin^2 \theta)$.

17. $\sin(\theta + 30°) \cos(\theta + 60°) = \frac{1}{4}(\cos^2 \theta - 3 \sin^2 \theta)$.

18. Given $\sin\theta = \frac{3}{5}$, $\sin\phi = \frac{5}{13}$, θ and ϕ both acute. Find
(a) $\sin(\theta+\phi)$, (b) $\cos(\theta+\phi)$, (c) $\tan(\theta+\phi)$,
(d) $\cot(\theta+\phi)$, (e) $\sin(\theta-\phi)$, (f) $\cos(\theta-\phi)$,
(g) $\tan(\theta-\phi)$, (h) $\cot(\theta-\phi)$, (i) $\sin(\phi-\theta)$,
(j) $\cos(\phi-\theta)$, (k) $\tan(\phi-\theta)$, (l) $\cot(\phi-\theta)$.

19. Given $\sin\theta = \frac{8}{17}$, $\tan\phi = \frac{9}{40}$, t in quadrant II, ϕ in quadrant III. Find
(a) $\sin(\theta+\phi)$, (b) $\cos(\theta+\phi)$, (c) $\tan(\theta+\phi)$,
(d) $\cot(\theta+\phi)$, (e) $\sin(\theta-\phi)$, (f) $\cos(\theta-\phi)$,
(g) $\tan(\theta-\phi)$, (h) $\cot(\theta-\phi)$.

20. Given $\cos\theta = -\frac{4}{5}$, $\sin\phi = \frac{7}{25}$, θ in quadrant II. Find all possible values of the following:
(a) $\sin(\theta+\phi)$, (b) $\cos(\theta+\phi)$, (c) $\tan(\theta+\phi)$,
(d) $\cot(\theta+\phi)$, (e) $\sin(\theta\quad\phi)$, (f) $\cos(\theta-\phi)$,
(g) $\tan(\theta-\phi)$, (h) $\cot(\theta-\phi)$.

21. Given $\tan\theta = \frac{8}{15}$, $\cot\phi = \frac{12}{5}$. Find all possible values of
(a) $\sin(\theta+\phi)$, (b) $\cos(\theta+\phi)$, (c) $\tan(\theta+\phi)$,
(d) $\cot(\theta+\phi)$, (e) $\sin(\theta-\phi)$, (f) $\cos(\theta-\phi)$,
(g) $\tan(\theta-\phi)$, (h) $\cot(\theta-\phi)$.

Prove:

22. $$\sin(\theta+\phi+\psi) = \sin\theta\cos\phi\cos\psi + \cos\theta\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi - \sin\theta\sin\phi\sin\psi.$$

23. $$\cos(\theta+\phi+\psi) = \cos\theta\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi - \sin\theta\cos\phi\sin\psi - \sin\theta\sin\phi\cos\psi.$$

24. $$\tan(\theta+\phi+\psi) = \frac{\tan\theta+\tan\phi+\tan\psi-\tan\theta\tan\phi\tan\psi}{1-\tan\phi\tan\psi-\tan\psi\tan\theta-\tan\theta\tan\phi}.$$

25. $$\cot(\theta+\phi+\psi) \qquad \frac{\cot\theta\cot\phi\cot\psi-\cot\theta-\cot\phi-\cot\psi}{\cot\phi\cot\psi+\cot\psi\cot\theta+\cot\theta\cot\phi-1}$$

73. Functions of twice an angle.

If, in formulas (5), (3), (7), (9) of section 72, we substitute θ for ϕ, we obtain the following results:

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta,$$

or
$$\sin 2\theta = 2 \sin \theta \cos \theta; \tag{1}$$
$$\cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta,$$
or
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta; \tag{2}$$
$$\tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta},$$
or
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}; \tag{3}$$
$$\cot(\theta + \theta) = \frac{\cot \theta \cot \theta - 1}{\cot \theta + \cot \theta},$$
or
$$\cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta}. \tag{4}$$

Two other useful formulas for $\cos 2\theta$ may be derived as follows: Remembering that

$$\sin^2 \theta = 1 - \cos^2 \theta, \qquad \cos^2 \theta = 1 - \sin^2 \theta,$$

and substituting these separately in (2), we get

$$\cos 2\theta = 2 \cos^2 \theta - 1, \tag{5}$$
and
$$\cos 2\theta = 1 - 2 \sin^2 \theta. \tag{6}$$

74. Functions of half an angle.

From the relation connecting sine and cosine, and the formula for the cosine of twice an angle, we have

$$\cos^2 \phi + \sin^2 \phi = 1, \tag{1}$$
$$\cos^2 \phi - \sin^2 \phi = \cos 2\phi. \tag{2}$$

Adding these two equations, we get

$$2 \cos^2 \phi = 1 + \cos 2\phi.$$

From this we get

$$\cos^2 \phi = \frac{1 + \cos 2\phi}{}$$

or

$$\cos \phi = \pm\sqrt{\frac{1 + \cos 2\phi}{}}$$

If ϕ is replaced by $\frac{1}{2}\theta$, this becomes

$$\boldsymbol{\cos \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 + \cos \theta}{}}} \tag{3}$$

By subtracting (2) from (1) and proceeding as before, we obtain the formula

$$\sin \phi = \pm\sqrt{\frac{1 \quad \cos 2\phi}{}}$$

which is equivalent to

$$\boldsymbol{\sin \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 \quad \cos \theta}{}}} \tag{4}$$

The sign to be used in the foregoing formulas depends upon the quadrant in which $\frac{1}{2}\theta$ lies.

Dividing (4) by (3), we get

$$\tan \tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos}{1 + \cos \theta}} \tag{5}$$

Multiplying numerator and denominator of the right-hand side of this last equation by $\sqrt{1 - \cos \theta}$, we get

$$\tan \tfrac{1}{2}\theta = \frac{1 - \cos \theta}{\pm\sqrt{1 - \cos^2}}$$

or

$$\boldsymbol{\tan \tfrac{1}{2}\theta = \frac{1 - \cos \theta}{\sin \theta}}. \tag{6}$$

Here the ambiguous sign ($\pm$) is not needed. For the numerator of the fraction in (6) is always positive (or zero),

so that the sign of the right-hand member depends upon the denominator, namely, $\sin\theta$. Now $\sin\theta$ will be positive if θ is in either of the first two quadrants, and negative otherwise. But if θ is in quadrant I or quadrant II, $\frac{1}{2}\theta$ will be in quadrant I; if θ is in quadrant III or quadrant IV, $\frac{1}{2}\theta$ will be in quadrant II. However, $\tan\frac{1}{2}\theta$ will be positive if $\frac{1}{2}\theta$ is in quadrant I, negative if $\frac{1}{2}\theta$ is in quadrant II. Thus, $\sin\theta$ and $\tan\frac{1}{2}\theta$ will always have the same sign, and there is no ambiguity.

If we multiply both numerator and denominator of the fraction in (5) by $\sqrt{1 \mp \cos\theta}$, and reduce, we get

$$\tan\tfrac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta}, \tag{7}$$

where again there is no ambiguity.

Similarly, we obtain the formulas

$$\cot\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \tag{8}$$

$$\cot\tfrac{1}{2}\theta = \frac{\sin\theta}{1-\cos\theta}, \tag{9}$$

$$\cot\tfrac{1}{2}\theta = \frac{1+\cos\theta}{\sin\theta}. \tag{10}$$

EXERCISES VIII. D

1. Verify the formulas for $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\cot 2\theta$ by setting $\theta = 30°$.

2. Verify the formulas for $\sin 2\theta$, $\cos 2\theta$, $\cot 2\theta$ by setting $\theta = 45°$.

3. Find $\sin 120°$, $\cos 120°$, $\tan 120°$, $\cot 120°$ by using the functions of $60°$.

4. Verify the formulas for $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$, $\tan\frac{1}{2}\theta$, $\cot\frac{1}{2}\theta$ by setting $\theta = 60°$.

5. Find $\sin 15°$, $\cos 15°$, $\tan 15°$, $\cot 15°$ by setting $\theta = 30°$ in the formulas for the functions of $\frac{1}{2}\theta$.

6. Given $\cos\theta = \frac{24}{25}$, θ an acute angle. Find
(a) $\sin 2\theta$, (b) $\cos 2\theta$, (c) $\tan 2\theta$, (d) $\cot 2\theta$,
(e) $\sin\frac{1}{2}\theta$, (f) $\cos\frac{1}{2}\theta$, (g) $\tan\frac{1}{2}\theta$, (h) $\cot\frac{1}{2}\theta$.
7. Given $\sin\theta = \frac{40}{41}$. Find
(a) $\sin 2\theta$, (b) $\cos 2\theta$, (c) $\tan 2\theta$, (d) $\cot 2\theta$,
(e) $\sin\frac{1}{2}\theta$, (f) $\cos\frac{1}{2}\theta$, (g) $\tan\frac{1}{2}\theta$, (h) $\cot\frac{1}{2}\theta$.
8. Given $\tan\theta = -2$. Find
(a) $\sin 2\theta$, (b) $\cos 2\theta$, (c) $\tan 2\theta$, (d) $\cot 2\theta$,
(e) $\sin\frac{1}{2}\theta$, (f) $\cos\frac{1}{2}\theta$, (g) $\tan\frac{1}{2}\theta$, (h) $\cot\frac{1}{2}\theta$.

Prove the following identities:

9. $\tan(45° + \frac{1}{2}\theta) = \dfrac{1 + \cos\theta + \sin\theta}{1 + \cos\theta - \sin\theta}$.

10. $\sin\theta = \dfrac{2\tan\frac{1}{2}\theta}{1 + \tan^2\frac{1}{2}\theta}$.

11. $\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta = 2\csc\theta$.

12. A picture of height 5 feet hangs on the wall, with its lower edge 4 feet from the floor. At a certain point on the floor, directly in front of the picture, the angle subtended by the picture (that is, by its vertical dimension of 5 feet) is equal to the angle of elevation of the lower edge of the picture. How far is this point from the wall?

Prove:

13. $\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta = \pm\sqrt{1 + \sin\theta}$.
14. $\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta = \pm\sqrt{1 - \sin\theta}$.

75. Sums and differences of functions.

By the addition and subtraction formulas for the sine and cosine, we have

$$\sin(x + y) = \sin x\cos y + \cos x\sin y, \tag{1}$$

$$\sin(x - y) = \sin x\cos y - \cos x\sin y, \tag{2}$$

$$\cos(x + y) = \cos x\cos y - \sin x\sin y, \tag{3}$$

$$\cos(x - y) = \cos x\cos y + \sin x\sin y. \tag{4}$$

Addition of (1) and (2) gives

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y. \tag{5}$$

If we let

$$x + y = \theta, \qquad x - y = \phi, \tag{6}$$

and solve for x and y we find that

$$x = \tfrac{1}{2}(\theta + \phi), \qquad y = \tfrac{1}{2}(\theta - \phi). \tag{7}$$

Thus, (5) becomes

$$\boldsymbol{\sin \theta + \sin \phi = 2 \sin \tfrac{1}{2}(\theta + \phi) \cos \tfrac{1}{2}(\theta - \phi).} \tag{8}$$

Subtracting (2) from (1) gives

$$\sin(x + y) - \sin(x - y) = 2 \cos x \sin y,$$

which, by the substitutions (6) and (7), becomes

$$\boldsymbol{\sin \theta - \sin \phi = 2 \cos \tfrac{1}{2}(\theta + \phi) \sin \tfrac{1}{2}(\theta - \phi).} \tag{9}$$

From (3) and (4) we obtain, in a similar manner,

$$\boldsymbol{\cos \theta + \cos \phi = 2 \cos \tfrac{1}{2}(\theta + \phi) \cos \tfrac{1}{2}(\theta - \phi),} \tag{10}$$

$$\boldsymbol{\cos \theta - \cos \phi = -2 \sin \tfrac{1}{2}(\theta + \phi) \sin \tfrac{1}{2}(\theta - \phi).} \tag{11}$$

EXERCISES VIII. E

Represent as a product:

1. $\sin 40° + \sin 20°$.

2. $\cos 80° - \cos 20°$.

3. $\cos 60° + \cos 40°$.

4. $\sin 30° - \sin 80°$.

5. $\cos 38° + \cos 42°$.

6. $\sin 35° + \sin 25°$.

7. $\sin 40° + \sin 25°$.

8. $\cos 17° - \cos 36°$.

9. $\sin 32° + \cos 22°$.

SUGGESTION. $\cos 22° = \sin(90° - 22°)$.

10. $\cos 10° + \sin 17°$.

11. $\sin 44° + \cos 40°$.

12. $\sin 4\theta - \sin 2\theta$.

13. $\sin 3\theta + \sin \theta$.

14. $\cos 5\theta + \cos 9\theta$.

15. $\sin \frac{1}{2}\theta + \sin\theta$.

16. $\cos 7\theta - \cos 3\theta$.

17. $\cos 4\theta + \cos 3\theta$.

Prove:

18. $\sin\theta + \cos\theta = \sqrt{2}\cos(\theta - 45°)$.

SUGGESTION. $\cos\theta = \sin(90° - \theta)$.

19. $\dfrac{\sin\theta + \sin\phi}{\cos\theta - \cos\phi} = \cot\tfrac{1}{2}(\phi - \theta)$.

20. $\dfrac{\sin\theta - \sin\phi}{\sin\theta + \sin\phi} = \dfrac{\tan\frac{1}{2}(\theta - \phi)}{\tan\frac{1}{2}(\theta + \phi)}$.

21. $\dfrac{\sin 3\theta + \sin 5\theta}{\cos 3\theta - \cos 5\theta} = \cot\theta$.

22. $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \dfrac{\sqrt{3}}{3}$.

23. $\cos 20° + \cos 100° + \cos 140° = 0$.

24. $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 4\cos\theta\cos 2\theta\sin 4\theta$.

25. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta$.

MISCELLANEOUS EXERCISES VIII. F

Prove:

1. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
2. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
3. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.
4. $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$.
5. $\sin 4\theta = 4\sin\theta\cos\theta(1 - 2\sin^2\theta)$
6. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.
7. $\tan 4\theta = \dfrac{4\tan\theta(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$.
8. $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot\theta(\cot^2\theta - 1)}$
9. $\tan\theta + \tan\phi = \dfrac{\sin(\theta + \phi)}{\cos\theta\cos\phi}$.
10. $\tan\theta - \tan\phi = \dfrac{\sin(\theta - \phi)}{\cos\theta\cos\phi}$.
11. $\cot\theta + \cot\phi = \dfrac{\sin(\theta + \phi)}{\ldots}$.

12. $\cot\theta - \cot\phi = \dfrac{\sin(\phi - \theta)}{\sin\theta\sin\phi}$.

13. $\dfrac{\sin\theta + \sin\phi}{\cos\theta + \cos\phi} = \tan\frac{1}{2}(\theta + \phi)$.

14. $\dfrac{\cos\theta - \cos\phi}{\cos\theta + \cos\phi} = -\tan\frac{1}{2}(\theta + \phi)\tan\frac{1}{2}(\theta - \phi)$.

15. $\dfrac{\cos(n-2)\theta - \cos n\theta}{\sin(n-2)\theta + \sin n\theta} = \tan\theta$.

16. $\sin^2\theta - \sin^2\phi = \sin(\theta + \phi)\sin(\theta - \phi)$.

17. $\cos^2\theta - \cos^2\phi = -\sin(\theta + \phi)\sin(\theta - \phi)$.

18. $\dfrac{\sin(\theta + \phi)}{\sin(\theta - \phi)} = \dfrac{\tan\theta + \tan\phi}{\tan\theta - \tan\phi} = \dfrac{\cot\phi + \cot\theta}{\cot\phi - \cot\theta}$.

19. $\dfrac{\cos(\theta + \phi)}{\sin(\theta - \phi)} = \dfrac{1 - \tan\theta\tan\phi}{\tan\theta - \tan\phi} = \dfrac{1 - \cot\theta\cot\phi}{\cot\theta - \cot\phi}$

20. $\dfrac{3\sin\theta - \sin 3\theta}{3\cos\theta + \cos 3\theta} = \tan^3\theta$.

21. $\sin\theta + \sin 3\theta + \sin 5\theta = \dfrac{\sin^2 3\theta}{\sin\theta}$.

22. Given $\sin\theta = \frac{4}{5}$, $\cos\phi = \frac{12}{13}$, both angles acute. Find
(a) $\sin(\theta+\phi)$, (b) $\cos(\theta+\phi)$, (c) $\tan(\theta+\phi)$, (d) $\cot(\theta+\phi)$,
(e) $\sin(\theta-\phi)$, (f) $\cos(\theta-\phi)$, (g) $\tan(\theta-\phi)$, (h) $\cot(\theta-\phi)$,
(i) $\sin 2\theta$, (j) $\cos 2\theta$, (k) $\tan 2\theta$, (l) $\cot 2\theta$,
(m) $\sin\frac{1}{2}\theta$, (n) $\cos\frac{1}{2}\theta$, (o) $\tan\frac{1}{2}\theta$, (p) $\cot\frac{1}{2}\theta$,
(q) $\sin\frac{1}{2}\phi$, (r) $\cos\frac{1}{2}\phi$, (s) $\tan\frac{1}{2}\phi$, (t) $\cot\frac{1}{2}\phi$,
(u) $\sin\theta + \sin\phi$, (v) $\sin\theta - \sin\phi$,
(w) $\cos\theta + \cos\phi$, (x) $\cos\theta - \cos\phi$.

23. Given $\tan\theta = \frac{7}{24}$, $\cos\phi = -\frac{40}{41}$. Find all possible values for the expressions (a)–(x) in the preceding exercise.

24. Find $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, $\tan 22\frac{1}{2}°$, $\cot 22\frac{1}{2}°$ by using the known functions of 45°.

25. Find sin 18°.

SOLUTION. Let $\theta = 18°$; then $3\theta = 54° = 90° - 2\theta$.

$$\cos 3\theta = \cos(90° - 2\theta) = \sin 2\theta.$$

Using exercise 2 above, we get

$$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta,$$

or $$\cos\theta(4\cos^2\theta - 2\sin\theta - 3) = 0.$$

Setting the first factor equal to zero, we get

$$\cos\theta = 0, \qquad \theta = 90^\circ \text{ (not } 18^\circ\text{)},$$

and this value must be discarded. From the second factor we get, after a slight reduction,

$$4\sin^2\theta + 2\sin\theta - 1 = 0.$$

This quadratic equation yields

$$\sin\theta = \frac{-1 \pm \sqrt{5}}{4}.$$

Since $\sin\theta$ must here be positive, we retain the upper sign only, and write

$$\sin 18^\circ = \frac{-1 + \sqrt{5}}{4}.$$

26. Find $\cos 18^\circ$, $\tan 18^\circ$, $\cot 18^\circ$.

27. Find $\sin 36^\circ$, $\cos 36^\circ$, $\tan 36^\circ$, $\cot 36^\circ$.

28. Find $\sin 9^\circ$, $\cos 9^\circ$.

29. Find $\sin 3^\circ$, $\cos 3^\circ$.

30. Find $\sin 6^\circ$, $\cos 6^\circ$.

31. A flagpole 34 feet high stands on top of a tower 30 feet high. From a certain point in the same horizontal plane with the base of the tower, the angle subtended by the pole is equal to the angle of elevation of the top of the tower. Find the distance from this point to the base of the tower.

32. A tree stands on the edge of a small lake. A man stands on the opposite side of the lake, his eye being at a height h above the foot of the tree. He finds that the angle of elevation of the top of the tree is E and the angle of depression of its reflection in the water is D. Show that the height of the tree is

$$\frac{h\sin(D + E)}{\sin(D - E)}.$$

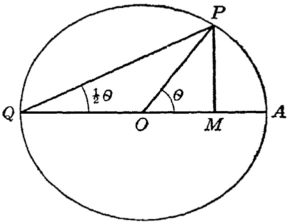


FIG. 64

33. The radius of the circle in Fig. 64 is 1. Consequently MP

$= \sin \theta$, $OM = \cos \theta$. Prove that $AQP = \frac{1}{2}\theta$, and show how to obtain the functions of $\frac{1}{2}\theta$ from the figure.

34. Draw a similar figure for the case in which θ is obtuse, and show that the same method applies.

Prove that if A, B, C are the angles of a triangle, then

35. $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$

36. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$

37. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

38. $\sin A + \sin B - \sin C = 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C.$

39. $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C.$

40. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$

41. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$

42. $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C.$

43. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C.$

44. $\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C).$

45. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$

46. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C.$

47. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$

48. $\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C = 1 - 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$

49. $\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B - \sin^2 \frac{1}{2}C = 1 - 2 \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C.$

50. $\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C.$

51. $\tan \frac{1}{2}A \tan \frac{1}{2}B + \tan \frac{1}{2}B \tan \frac{1}{2}C + \tan \frac{1}{2}C \tan \frac{1}{2}A = 1.$

52. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$

53. $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C)$
$= 4 \sin A \sin B \sin C.$

54. $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B)$
$= 4 \sin \frac{1}{2}(B - C) \sin \frac{1}{2}(C - A) \sin \frac{1}{2}(A - B).$

55. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$

56. Prove the law of tangents by using the law of sines and (8) and (9) of section 75.

SUGGESTION. From the law of sines we get

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

★76. Reduction of $a \cos \theta \pm b \sin \theta$.

It is frequently desirable to reduce an expression of the form $a \cos \theta \pm b \sin \theta$ to the form

$r \sin(\theta \pm \phi)$ or $r \cos(\theta \pm \phi)$.

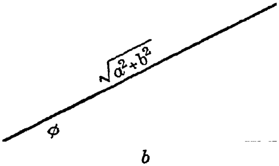


FIG. 65

These transformations adapt the expressions to logarithmic computations, and are often of advantage in solving trigonometric equations. They may be made in the following manner:

$$a \cos \theta + b \sin \theta$$
$$= \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta\right).$$

Let us introduce an angle ϕ such that (see Fig. 65)

$$\sin \phi \qquad a \qquad\qquad \cos \phi = \qquad b$$

Then,

$$a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2}\,(\cos \theta \cos \phi + \sin \theta \sin \phi)$$
$$= \sqrt{a^2 + b^2} \cos(\theta - \phi).$$

Example.

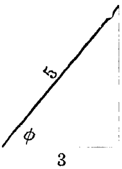


FIG. 66

Reduce $3 \cos \theta - 4 \sin \theta$ to the form $r \cos(\theta + \phi)$.

SOLUTION. Multiply and divide by $\sqrt{3^2 + 4^2} = 5$:

$$3 \cos \theta - 4 \sin \theta = 5(\tfrac{3}{5} \cos \theta - \tfrac{4}{5} \sin \theta).$$

If ϕ is an angle such that (see Fig. 66),

$$\cos \phi = \frac{}{5}, \qquad \sin \phi = \frac{}{5},$$

then

$$3 \cos \theta - 4 \cos \theta = 5(\cos \theta \cos \phi - \sin \theta \sin \phi)$$
$$= 5 \cos(\theta + \phi).$$

From tables we find $\phi = 53°$ approximately. Therefore,

$$3 \cos \theta - 4 \sin \theta = 5 \cos(\theta + 53°).$$

EXERCISES VIII. G

1. Reduce $\sin \theta - \cos \theta$ to the form $r \sin(\theta - \phi)$, and find the angle ϕ.

2. Reduce $\sin \theta + 2 \cos \theta$ to the form $r \sin(\theta + \phi)$, and find ϕ.

Reduce each of the following expressions to one of the forms $r \cos(\theta \pm \phi)$, $r \sin(\theta \pm \phi)$, and find the value of ϕ.

3. $12 \cos \theta - 5 \sin \theta$.

4. $3 \sin \theta - 2 \cos \theta$.

5. $\cos \theta + \sqrt{3} \sin \theta$.

6. $\frac{1}{2} \sin \theta + \dfrac{\sqrt{3}}{2} \cos \theta$.

7. $\cos \theta + \sin \theta$.

8. $0.4 \cos \theta + 1.5 \sin \theta$.

9. $0.3642 \cos \theta - 1.2476 \sin \theta$.

SUGGESTION. Use logarithms.

10. Given $3 \sin \theta - 4 \cos \theta = 2$. Reduce to the form $r \sin(\theta - \phi) = 2$, in which r and ϕ are known. Find $\sin(\theta - \phi)$, and, from tables, $\theta - \phi$. Finally, find a value of θ which satisfies the original equation.

CHAPTER IX

Radian Measure

77. Radian

One **radian** is the measure of an angle which, if its vertex is placed at the center of a circle, intercepts on the circumference an arc equal in length to the radius. It may be abbreviated 1 rad. or $1^{(r)}$. This unit of measurement of angle is important in deriving and in simplifying certain formulas in calculus and higher mathematics. Radian measure is sometimes called **circular measure** of angles.

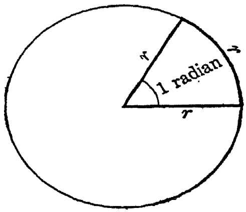


FIG. 67

78. Relation between radian and degree.

The relation between the radian and the degree may be found as follows: The circumference of a circle is 2π times the radius. Therefore, the number of radians in 360° is 2π. That is, $360° = 2\pi^{(r)}$. If we divide this equation by 2 we get

$$\mathbf{180° = \pi^{(r)} = 3.1416^{(r)}.} \tag{1}$$

This is a convenient relation to remember when reducing degrees to radians or radians to degrees.

Frequently used are the following angles:

$$90° = \frac{\pi}{2}^{(r)} \qquad 60° = \frac{\pi}{3}^{(r)} \qquad 45° = \frac{\pi}{4}^{(r)} \qquad 30° = \frac{\pi}{6}^{(r)}$$

From (1) we get

$$1^\circ = \frac{\pi^{(r)}}{180} = 0.017453^{(r)},$$

also

$$1^{(r)} \quad \frac{180^\circ}{\pi} = 57.29578^\circ = 57^\circ\ 17'\ 44.8''.$$

Example 1.

Convert $37^\circ\ 43'\ 26''$ to radians.

SOLUTION. $37^\circ\ 43'\ 26'' = 37.7239^\circ$
$= 37.7239 \times 0.017453^{(r)} = 0.6584^{(r)}$.

Example 2.

Convert 2.25 radians to degrees, minutes, and seconds.

SOLUTION. $2.25^{(r)} = 2.25 \times 57.29578^\circ$
$= 128.9155^\circ = 128^\circ\ 54'\ 56''$.

If tables for converting degrees to radians (e.g., Table IV of the Macmillan Logarithmic and Trigonometric Tables) and radians to degrees (e.g., Table Va of the Macmillan Tables) are available, problems such as the foregoing are considerably simplified.

EXERCISES IX. A

1. Reduce the following angles to radians, giving the results in terms of π:

(a) 10°, (b) 35°, (c) 48°, (d) 70°,
(e) 150°, (f) 280°, (g) 18°, (h) 400°,
(i) $10^\circ\ 30'$, (j) $24^\circ\ 45'$, (k) $480^\circ\ 45'$, (l) $17^\circ\ 20'$.

2. Reduce the following angles to radians, giving the results in decimal form:

(a) 15°, (b) $10^\circ\ 17'$, (c) $10^\circ\ 17'\ 22''$,
(d) $18^\circ\ 24'\ 16''$, (e) $370^\circ\ 15'\ 8''$, (f) $142^\circ\ 25'\ 30''$,
(g) $67^\circ\ 43'\ 52''$, (h) $21^\circ\ 21'\ 21''$, (i) $2^\circ\ 3'\ 49''$.

3. Express the following angles in degrees. (When it is quite clear that radian measure is to be used, the symbol for radians

is commonly omitted. Thus, the angle π radians may be written simply π.)

(a) $\frac{\pi}{10}$, (b) $\frac{\pi}{12}$, (c) $\frac{\pi}{15}$, (d) $\frac{\pi}{18}$,
(e) $\frac{2\pi}{3}$, (f) $\frac{3\pi}{4}$, (g) $\frac{3\pi}{2}$, (h) $\frac{5\pi}{6}$,
(i) $\frac{\pi}{5}$, (j) $\frac{2\pi}{5}$, (k) $\frac{3\pi}{5}$, (l) $\frac{4\pi}{5}$,
(m) $\frac{3\pi}{10}$, (n) $\frac{7\pi}{15}$, (o) $\frac{5\pi}{12}$, (p) $\frac{7\pi}{9}$.

4. Express the following angles in degrees, minutes. and seconds:

(a) $\frac{\pi}{8}$, (b) $\frac{\pi}{50}$, (c) $\frac{\pi}{150}$, (d) $\frac{\pi}{7}$,
(e) $\frac{2\pi}{11}$, (f) $\frac{\pi}{40}$, (g) $\frac{5\pi}{24}$, (h) $\frac{\pi}{16}$,
(i) $\frac{\pi}{25}$, (j) $\frac{11\pi}{50}$, (k) $\frac{3\pi}{32}$, (l) $\frac{\pi}{48}$.

5. Reduce to degrees, minutes, and seconds:

(a) $\frac{1}{2}^{(r)}$, (b) $\frac{2}{3}^{(r)}$, (c) $\frac{2}{7}^{(r)}$, (d) $2\frac{5}{6}^{(r)}$,
(e) $3.2^{(r)}$, (f) $1.236^{(r)}$, (g) $0.1236^{(r)}$, (h) $0.1236\pi^{(r)}$.

6. One angle of a triangle is 25°, another angle is 1.3 radians. Find the third angle in degrees, and also in radians.

7. Find, in radians, the angle between the hands of a clock at (a) 2 o'clock, (b) 5 o'clock, (c) 7:30, (d) 5:15.

8. Through how many radians does the hour hand of a watch turn in (a) 5 hours? (b) $\frac{1}{2}$ hour? (c) 10 minutes? (d) 3 days? (e) between 8:00 a.m. and 5:30 p.m.?

9. Through how many radians does the earth turn in (a) 1 hour? (b) 1 minute? (c) 3 hours and 20 minutes? (d) 3 days? (e) between 8:00 a.m. and 5:30 p.m.?

10. An automobile wheel is 2 feet in diameter. Through how many radians does it turn while the automobile travels 1 mile?

11. Find the value of each of the following functions, using tables if necessary:

(a) $\sin\frac{\pi}{3}$, (b) $\cos\frac{2\pi}{3}$, (c) $\tan\frac{5\pi}{4}$,
(d) $\cot\left(-\frac{\pi}{6}\right)$, (e) $\sec\frac{3\pi}{4}$, (f) $\csc\frac{5\pi}{6}$,

(g) $\sin \frac{3\pi}{2}$,	(h) $\cos \frac{2\pi}{9}$,	(i) $\tan \frac{21\pi}{20}$;
(j) $\cot \frac{6\pi}{7}$,	(k) $\sin \frac{12\pi}{11}$,	(l) $\cos \frac{\pi}{13}$,
(m) $\sin 1^{(r)}$,	(n) $\cos 2.3^{(r)}$,	(o) $\tan(-5.2)^{(r)}$,
(p) $\cot 0.435^{(r)}$,	(q) $\sin 0.01^{(r)}$,	(r) $\cos 100^{(r)}$.

79. Relation between arc and angle.

Suppose that the arc CD in Fig. 68 subtends a central angle of θ radians, and that the arc AB subtends a central angle of 1 radian. Since central angles have the same ratio as their intercepted arcs, $\theta/1 = s/r$, or

$$\theta = \frac{s}{r}, \qquad s = r\theta. \tag{1}$$

That is,

$$arc = radius \times angle \text{ (in radians).}$$

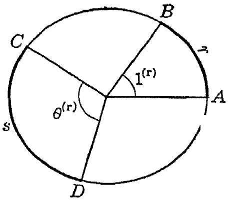


FIG. 68

It is readily seen that for a unit circle (that is, a circle whose radius is 1), a central angle expressed in radians is numerically equal to the intercepted arc expressed in linear units. For example, in a circle having a radius of 1 inch, a central angle of 2.3 radians will intercept an arc of 2.3 inches.

Example

What is the length of the arc intercepted by a central angle of 95° in a circle whose radius is 12 feet?

SOLUTION. First reduce the angle to radians:

$$\theta = 95 \times \frac{\pi}{180} = 1.66.$$

From (1), $\qquad s = r\theta = 12 \times 1.66 = 19.9 \text{ ft.}$

*80. Angular velocity.

If a wheel turns completely round thirty times in a second, we say that it is rotating at the rate of thirty revo-

lutions per second, abbreviated r.p.s. (Similarly, the expression " revolutions per minute " is abbreviated r.p.m.) A spoke of this wheel will turn through 360° in each rotation, or through $30 \times 360° = 10800°$ per second. Since the spoke turns through 2π radians in each rotation, in each second it turns through $30 \times 2\pi$ radians, or 60π radians. The wheel is said to have an **angular velocity** of 30 r.p.s., or 10800° per second, or 60π radians per second.

Suppose now that the wheel has a radius of 2 feet. When the wheel has turned through an angle of 1 radian, a point on the circumference will have moved through 2 feet. For any number of radians through which the spoke turns, a point on the circumference travels twice that number of feet. But the wheel turns through 60π radians per second. Hence, a point on the circumference moves through $60\pi \times 2$ feet per second, or it has a **linear velocity** of 120π feet per second.

In general, let us suppose that a line OP, of length r, is rotating about the point O with a constant angular velocity. If it turns through an angle θ in t units of time, the angular velocity ω is given by the formula

$$\boldsymbol{\omega = \frac{\theta}{t},}$$

from which we get

$$\boldsymbol{\theta = \omega t.} \tag{1}$$

Since the length of OP is r, we find from (1) of the preceding section that the arc through which P moves while OP turns through θ radians is

$$s = r\theta = r\omega t. \tag{2}$$

But if v is the velocity of P in linear units per unit of time, we have $s = vt$, that is,

$$vt = r\omega t.$$

Dividing by t, we obtain the formula

$$v = r\omega. \tag{3}$$

Example.

A rotating wheel has a radius of 2 feet 6 inches. A point on the rim of the wheel moves 10 yards in 3 seconds. Find the angular velocity of the wheel.

SOLUTION. The linear velocity of the point on the rim is

$$\frac{10}{3} \text{ yd. per sec.} = \frac{30}{3} \text{ ft. per sec.} = 10 \text{ ft. per sec.}$$

(It should be noted that like quantities must be reduced to the same unit.) Substituting $v = 10$, $r = 2.5$ in (3), we get

$$10 = 2.5\omega, \qquad \omega = \frac{10}{2.5} = 4^{(r)} \text{ per sec.}$$

EXERCISES IX. B

1. A central angle in a circle of radius 10 inches intercepts an arc of 14 inches. How many radians are there in the angle?

2. A circle has a radius of 15 inches. Find, in radians, a central angle subtended by an arc of (a) 25 inches, (b) 1 inch, (c) 2 feet 6 inches.

3. An arc of 4 feet 3 inches subtends a central angle of 1.2 radians. Find the radius of the circle.

4. Find the length of the arc intercepted by an inscribed angle of 0.35 radian in a circle whose radius is 3 inches.

5. The angle between a tangent and a chord is $\frac{1}{4}$ radian. If the length of the arc subtended by the chord is 5 inches, what is the radius of the circle?

6. Find, in radians, the angle between the tangents to a circle at two points whose distance apart, measured on the circumference of the circle, is 350 feet, the radius of the circle being 800 feet.

7. Each of two tangents from an external point to a circle is 3 inches long. The smaller arc which they intercept is 2 radians. Find the radius of the circle.

8. A flywheel 1.5 feet in diameter has an angular velocity of 8 radians per second. Find the linear velocity of a point on the rim.
9. The wheel of an automobile is 2 feet in diameter. The automobile is traveling at the rate of 30 miles an hour. Find the angular velocity of the wheel in radians per minute.
10. A belt travels around two pulleys whose diameters are 10 inches and 4 feet respectively. The larger pulley makes 100 revolutions per minute. Find the angular velocity of the smaller pulley in radians per second.
11. An airplane propeller measures 8 feet from tip to tip. It rotates at the rate of 1800 r.p.m. (a) Find its angular velocity in radians per second. (b) Find the linear speed of a point on the tip of one of the blades, assuming that the airplane itself is not in motion.

★81. Area of sector and of segment.

A **sector** of a circle is a portion of the circle bounded by two radii and their intercepted arc. In plane geometry it is shown that the area of a sector is equal to one-half its arc times the radius of the circle. Thus, the area of the sector OAB in Fig. 69 is given by the formula $\frac{1}{2}rs$, s being the length of the arc AB. If the angle θ in this figure is expressed in radians, we have $s = r\theta$, and, substituting this in the expression $\frac{1}{2}rs$, we have

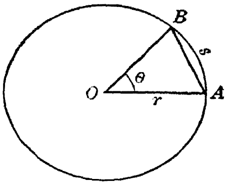


Fig. 69

$$\text{area of sector} = \tfrac{1}{2}r^2\theta \qquad (\theta \text{ in radians}). \tag{1}$$

A **segment** of a circle is a portion of the circle bounded by an arc and its chord. The area of the sector bounded by arc AB and chord AB in Fig. 69 is obviously equal to the area of the sector AOB minus the area of the triangle AOB. But the area of the triangle is equal to $\frac{1}{2}r^2 \sin \theta$. (See section 51.) Thus,

$$\text{area of segment} = \tfrac{1}{2}r^2(\theta - \sin \theta) \qquad (\theta \text{ in radians}). \tag{2}$$

EXERCISES IX. C

1. Find the area of a sector having an angle of 0.75 radian in a circle whose radius is 6 inches. Find the area of the corresponding segment.
2. The perimeter of a circular sector, whose angle is 1.5 radians, is 14 inches. Find the radius of the circle.
3. The area of a sector of a circle, whose radius is 15 inches, is 135 square inches. Find the angle of the sector.
4. The area of a sector of a circle is 705.6 square centimeters. If the angle of the sector is 0.45 radian, what is the radius of the circle?
5. The central angle subtended by the arc of a segment of a circle is 1.3 radians. The area of the segment is 17 square inches. Find the radius of the circle.
6. A chord of 0.75 foot subtends an arc of 0.75 radian. Find the area of the segment bounded by the chord and the arc.
7. A segment of height 3 inches (distance from center of chord to center of arc) has an arc of $\frac{1}{3}$ radian. Find the area of the segment.
8. The perimeter of a segment of a circle is 22 inches. The arc is 2 radians. What is the area of the segment?
9. A right circular cone is made by cutting out a sector, whose angle is 1.2 radians, from a circular piece of paper of radius 5 inches, and then placing the cut edges of the remaining portion together. Find (a) the lateral area and (b) the volume of the cone. (Lat. area $= \frac{1}{2}$ circumf. of base $\times$ slant ht., Vol. $= \frac{1}{3}$ area of base $\times$ alt.)
10. Find the area of a $35°$ sector in a circle whose diameter is 7 inches. Find the area of the corresponding segment.
11. A horizontal cylindrical tank has a diameter of 4 feet and a length of 10 feet. It is filled with liquid to a depth of 8 inches. How many gallons of liquid does it contain? (1 gal. = 231 cu. in.)

★82. Angles near 0° or 90°.

For angles near 0° or 90° (say between 0° and 3° or between 87° and 90°) interpolation by proportional parts may yield results which are considerably in error.

This difficulty may be remedied, to considerable extent, by using special tables for such angles (e.g., Table IIIa of the Macmillan Logarithmic and Trigonometric Tables). However, the difficulty may be met in another way, which is also useful for still further refinements even if such special tables are available.

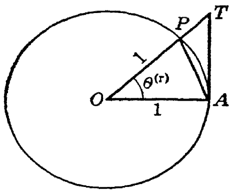


FIG. 70

In Fig. 70, AT is tangent to the unit circle with center at O, AP is a chord, angle θ is measured in radians. It is evident that, in area,

$$\text{triangle } AOP < \text{sector } AOP < \text{triangle } AOT. \qquad (1)$$

But by formula (7) of section 51,

$$\text{area triangle } AOP = \tfrac{1}{2}\sin\theta. \qquad (2)$$

By formula (1) of the preceding section,

$$\text{area sector } AOP = \tfrac{1}{2}\theta. \qquad (3)$$

Since $AT = \tan\theta$,

$$\text{area triangle } OAT = \tfrac{1}{2}\tan\theta. \qquad (4)$$

Substituting (2), (3), (4) in (1), and dividing through by $\frac{1}{2}$, we get

$$\sin\theta < \theta < \tan\theta. \qquad (5)$$

That is, if a positive acute angle is measured in radians, it will always be greater than its sine and less than its tangent.

If we divide (4) by $\sin\theta$, we find that

$$1 < \frac{\theta}{\sin\theta} < \sec\theta. \qquad (6)$$

Now, as the angle θ shrinks in size to 0, $\sec\theta$ approaches the value 1. It is evident, therefore, that as θ approaches 0, the

ratio $\theta/\sin\theta$ must also approach 1 as its value. This may be written

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1. \tag{7}$$

Similarly, we may divide (5) by $\tan\theta$, getting

$$\cos\theta < \frac{\theta}{\tan\theta} < 1. \tag{8}$$

Since $\cos 0 = 1$, it follows that

$$\lim_{\theta\to 0}\frac{\theta}{\tan\theta} = 1. \tag{9}$$

It may be noted that (7) and (9) are equivalent, respectively, to

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1, \qquad \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1. \tag{10}$$

These equations mean that

$$\sin\theta \approx \theta, \qquad \tan\theta \approx \theta \qquad (\theta \text{ small}), \tag{11}$$

where the symbol $\approx$ denotes "is approximately equal to." This may be verified by reference to tables. To illustrate,

$\sin 2° = 0.03490, \qquad \tan 2° = 0.03492, \qquad 2° = 0.03491^{(r)}.$

If θ is near $90°$ $\left(\text{i.e., } \frac{\pi}{2}^{(r)}\right)$, we may write $\theta = \frac{\pi}{2} - \phi$, and ϕ will be a small angle. Consequently,

$$\cos\theta = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi \approx \phi = \frac{\pi}{2} - \theta. \tag{12}$$

Similarly, $$\cot\theta \approx \frac{\pi}{2} - \theta. \tag{13}$$

We may summarize as follows:

If θ *is near* 0,

$$\begin{aligned} \sin\theta &\approx \tan\theta \approx \theta^{(r)}, \\ \cot\theta &\approx \csc\theta \approx \frac{1}{\theta^{(r)}}, \end{aligned} \tag{14}$$

$\cos\theta$ and $\sec\theta$ may be found from tables, as usual.

If θ *is near* 90° (*i.e.*, $\frac{\pi^{(r)}}{2}$),

$$\begin{aligned} \cos\theta &\approx \cot\theta \approx \frac{\pi}{2} - \theta^{(r)}, \\ \tan\theta &\approx \sec\theta \approx \frac{1}{\frac{\pi}{2} - \theta^{(r)}} \end{aligned} \tag{15}$$

$\sin\theta$ and $\csc\theta$ may be found from tables, as usual.

Example 1.

Find log tan 2′ 54″.

SOLUTION. 2′ 54″ = 0.048333° = $(0.048333 \times 0.017453)^{(r)}$.

$$\begin{aligned} \log 0.048333 &= 8.68425 - 10 \\ \log 0.017453 &= 8.24187 - 10 \\ \log \tan 2'\,54'' &= 6.92612 - 10 \end{aligned}$$

This agrees exactly with the value found in tables giving values for every second.

Example 2.

Find the angle subtended by a yardstick at a distance of 1 mile.

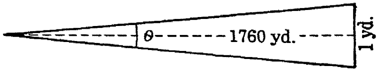


FIG. 71

SOLUTION. Strictly speaking, the yardstick would be the base of an isosceles triangle whose altitude is 1 mile, or 1760 yards. We could thus find (see Fig. 71)

$$\tan \tfrac{1}{2}\theta = \frac{0.5}{1760}$$

from which, since $\tan \frac{1}{2}\theta$ may be replaced by $\frac{1}{2}\theta$, θ is readily

obtainable. However, it makes no essential difference if we regard the yardstick as one side of a right triangle of which the other side is 1 mile. Indeed, probably the best way to regard the problem is to think of the yardstick as the arc, rather than the chord, of a circle of radius 1 mile. Any of these methods leads to the approximate equation

$$\theta = \frac{1}{1760} = 0.0005682^{(r)} = 1'\ 57.2''.$$

A slowly changing function does not determine the angle very definitely. For example, if it is given that $\log \cos \theta = 9.99990 - 10$, reference to a five-place table giving the values of the logarithmic functions for every minute, shows that θ may have any value from $1^\circ\ 12'$ to $1^\circ\ 15'$ inclusive. Hence we should, if possible, avoid using $\cos \theta$ if θ is near 0, or $\sin \theta$ if θ is near 90°.

EXERCISES IX. D

Find the values of the following functions:

1. (a) $\sin 1^\circ\ 13'\ 17''$,
(b) $\tan 1^\circ\ 13'\ 17''$,
(c) $\cot 1^\circ\ 13'\ 17''$.

2. (a) $\cos 89^\circ\ 2'\ 20''$,
(b) $\cot 89^\circ\ 2'\ 20''$,
(c) $\tan 89^\circ\ 2'\ 20''$.

3. (a) $\log \sin 54'\ 22''$,
(b) $\log \tan 54'\ 22''$,
(c) $\log \cot 54'\ 22''$

4. (a) $\log \cos 89^\circ\ 20'\ 54''$,
(b) $\log \cot 89^\circ\ 20'\ 54''$,
(c) $\log \tan 89^\circ\ 20'\ 54''$.

5. A railroad is inclined at an angle of $50'$ with the horizontal. How many feet does it rise in a horizontal distance of 2 miles?

6. A highway rises 70 feet in a horizontal distance of 1 mile. What is its angle of inclination?

7. If the moon is at a distance of 238860 miles from the earth, and its diameter subtends an angle of $31'\ 5''$ at the earth, what is its diameter?

8. If the sun is 92,897,000 miles from the earth, and subtends an angle of $31'\ 59''$ at the earth, what is its diameter?

9. At Alpha Centauri, the nearest star to our sun, the distance from the earth to the sun (see preceding exercise) subtends an angle of $0.76''$. Find the distance from the sun to the star.

10. The mean radius of the earth is approximately 3957 miles. It subtends an angle of 8.8″ at the sun. Find the distance from the earth to the sun.
11. If the mean radius of the earth (see preceding exercise) subtends an angle of 57′ 2.6″ at the moon, what is the distance from the earth to the moon?

Solve the following triangles:

12. $A = 1° 28.1'$,	$C = 90°$,	$a = 12.486$.
13. $C = 90°$,	$a = 0.76128$,	$b = 57.953$.
14. $A = 1° 13.2'$,	$B = 46° 21.4'$,	$a = 124.75$.
15. $a = 54321$,	$b = 28967$,	$c = 25422$.
16. $C = 56.9'$,	$a = 5.2389$,	$b = 1.9942$.
17. $B = 88° 15.3'$,	$C = 32° 19.7'$,	$a = 0.11654$.

*83. Mil.

A unit of angular measurement used in military science is the **mil,** which is $\frac{1}{1600}$ of a right angle, or $3'\ 22\frac{1}{2}''$. One degree is $17\frac{7}{9}$ mils. A mil is approximately equal to one thousandth of a radian (more accurately, 0.000982 radian). Practically, it is the angle subtended by a line of unit length at a distance of 1000 units.

If a line L units in length at a distance, or range, of R units, subtends an angle M (see Fig. 72), then the number of mils in M is given by the approximate formula

$$M \quad \frac{1000\,L}{R} \tag{1}$$

From this we get

$$L \approx 0.001\,RM, \qquad R \quad \frac{1000\,L}{M} \tag{2}$$

The errors resulting from the use of formulas (1) and (2) will be less than 2 per cent provided the angle is not greater than 680 mils (about 38°).

In Fig. 72, L is the base of an isosceles triangle whose vertex angle is M. If, as in Fig. 73, the lengths L and R

are the sides of a right triangle having the acute angle M opposite side L, formulas (1) and (2) still hold. In this

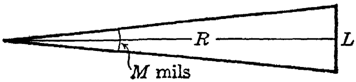


FIG. 72

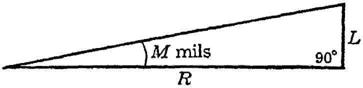


FIG. 73

case the error caused by using them will be less than 2 per cent if the angle is not greater than 340 mils (about 19°).

Example.

Find the angle subtended by an object 8 yards long at a distance of 2000 yards.

SOLUTION. Here $L = 8$, $R = 2000$, and from (1) we find

$$M \approx \frac{1000 \times 8}{20000} = 4 \text{ mils}.$$

EXERCISES IX. E

1. An object 20 feet long is 500 feet away. How many mils does it subtend if it is at right angles to the line of sight?
2. A tree 250 yards distant subtends an angle of 30 mils. How tall is it?
3. A boxcar which is known to be 42 feet long subtends an angle of 20 mils. If it is perpendicular to the line of vision, how far away is it?
4. A hill at a distance of 1560 meters subtends an angle of 40 mils. How high is it?
5. What angle does a pole 25 feet high subtend at a distance of 100 yards?
6. A balloon known to be 150 feet long is directly overhead and subtends an angle of 125 mils. How high is it?
7. A hill 50 meters high is 1500 meters away. At what angle with the horizontal must a gun be pointed in order for the projectile just to clear the top of the hill, if an allowance of 10 mils must be made for the fall of the projectile?
8. A tree 75 feet high is at a distance of 500 feet from a given point on the ground; 1500 feet farther away is a hill 350 feet

high. If a line is drawn from the point on the ground through the top of the tree, how far from the top of the hill will it strike?

9. A gun is 2500 yards from its target. A shot is fired and the projectile is observed to strike even with the target but 8 mils to the right. By how many yards did it miss the target?
10. Change into mils: $10°$, $15°$, $10'$, $10''$.
11. Change into degrees, minutes, and seconds: 10 mils, 50 mils, 100 mils.

CHAPTER X

Graphic Representations of the Trigonometric Functions

*84. Line representations of the trigonometric functions.

We shall now show how to represent the trigonometric functions by means of line segments. In so representing

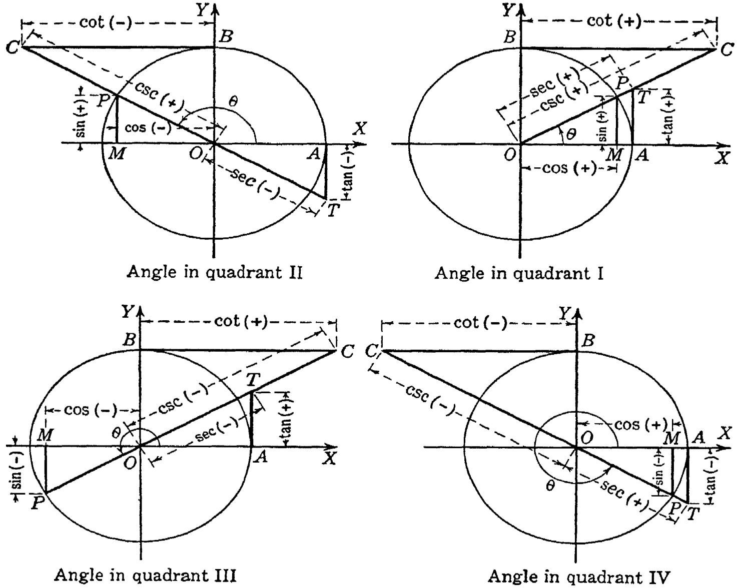


Fig. 74

the functions we shall make use of a **unit circle,** that is, a circle whose radius is 1.

The circles in Fig. 74 are unit circles. In this figure the

initial side of the angle θ is, as usual, in coincidence with the positive end of the x-axis; its terminal side is OP, P being the point in which the terminal side intersects the unit circle. Four different values of θ are shown, one in each of the four quadrants. In each case MP is drawn perpendicular to the x-axis, and the lines at A and B are tangent to the circle. (Points A and B are the intersections of the circle with the positive ends of the x- and y-axes respectively.)

Referring to the figure, we see that for θ in any quadrant,

$$\sin \theta = \frac{MP}{OP} \quad \frac{MP}{1} = MP,$$

$$\cos \theta = \frac{OM}{OP} \quad \frac{OM}{1} = OM.$$

The signs of these functions are determined by the directions of the segments MP and OP. The segment MP will be regarded as positive if the direction from M to P is upward, as negative if this direction is downward. The segment OM will be regarded as positive if the direction from O to M is to the right, as negative if this direction is to the left.

In order to complete this scheme of representing the functions, we must write the remaining functions as ratios in which the denominator is 1. This is accomplished by the selection of similar right triangles. Moreover, we wish to select the line segments which represent the functions so that they will have the proper signs.

To represent the tangent we note that

$$\tan \theta = \frac{MP}{OM} = \frac{AT}{OA} = \frac{AT}{1} = AT.$$

It is readily proved that the right triangles MOP and BOC are similar, and it follows that

$$\cot \theta = \frac{OM}{MP} - \frac{BC}{OB} = \frac{BC}{1} = BC.$$

The conventions regarding signs, as stated above, will apply to the segments AT and BC.

The secant and the cosecant are measured along the terminal side of the angle. We shall specify that when they are measured in the same direction as the terminal line, that is, from the origin out, they are positive, and when measured in the reverse direction they are negative. (Cf. section 72.) Then, from similar triangles, we have

$$\sec \theta = \frac{OP}{OM} = \frac{OT}{OA} = \frac{OT}{1} = OT,$$

$$\csc \theta = \frac{OP}{MP} = \frac{OC}{OB} = \frac{OC}{1} = OC.$$

It should be noted that the functions are not lines. They are ratios, and therefore abstract numbers. The values of the functions are given by the measures of the lengths of the lines (i.e., line segments) in terms of the radius as a unit. The use of the circle explains why the trigonometric functions are sometimes called **circular functions.** It also explains the origin of the terms " tangent " and " secant."

Certain relations connecting the functions can be proved very readily from Fig. 74. For example,

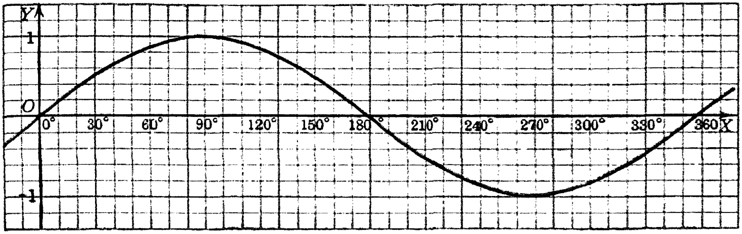
$$\sin^2 \theta + \cos^2 \theta = 1, \qquad 1 + \tan^2 \theta = \sec^2 \theta,$$
$$1 + \cot^2 \theta = \csc^2 \theta.$$

85. Graph of the sine.

A study of Fig. 74 shows that for an angle of 0° the line MP, representing the sine, disappears; that is, $\sin 0° = 0$. As the angle increases from 0°, the sine increases, until at 90° it reaches its maximum value of 1; as the angle increases further, the value of the sine decreases to 0 at 180°, and to -1 at 270°. It has now reached its minimum value, and as the angle increases beyond 270° the sine increases from -1 to 0, which value it reaches when the angle reaches 360°.

This variation in value of the sine is shown in Fig. 75,

which is the graph of $y = \sin x$. The values 1 and -1 are marked on the y-axis, and any convenient unit is chosen on the x-axis. The information of the preceding paragraph is supplemented by using tables to obtain values of y for a number of values of x, so that the points can be plotted



$y = \sin x$

FIG. 75

accurately. If a sufficient number of points are taken, a smooth curve can be drawn through them.

If tables are not conveniently at hand, the values of the sine for the angles 0°, 30°, 45°, 60°, 90°, 120°, and so on, can readily be calculated without tables. These values are listed in the accompanying table. From them the sine curve can often be plotted with sufficient accuracy.

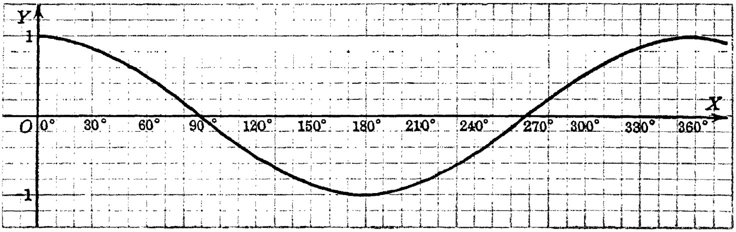
θ	$\sin\theta$	θ	$\sin\theta$
0°	0	180°	0
30°	$\frac{1}{2} = 0.50$	210°	$-\frac{1}{2} = -0.50$
45°	$\frac{\sqrt{2}}{2} = 0.71$	225°	$-\frac{\sqrt{2}}{2} = -0.71$
60°	$\frac{\sqrt{3}}{2} = 0.87$	240°	$-\frac{\sqrt{3}}{2} = -0.87$
90°	1	270°	-1
120°	$\frac{\sqrt{3}}{2} = 0.87$	300°	$-\frac{\sqrt{3}}{2} = -0.87$
135°	$\frac{\sqrt{2}}{2} = 0.71$	315°	$-\frac{\sqrt{2}}{2} = -0.71$
150°	$\frac{1}{2} = 0.50$	330°	$-\frac{1}{2} = -0.50$
180°	0	360°	0

These same angles are useful in constructing graphs of the other functions. (See following sections.)

If the angle increases beyond 360°, the sine runs through the same values again. Thus, the part of the graph between 0° and 360° is a complete pattern of the entire curve, which extends indefinitely both to the right and to the left. For this reason, 360° is called the **period** of the sine.

86. Graph of the cosine.

The cosine starts with its maximum value of 1 when the angle is 0°, decreases to 0 at 90°, to -1 at 180°, and then increases from this minimum value through 0 at 270° to 1



$y = \cos x$

FIG. 76

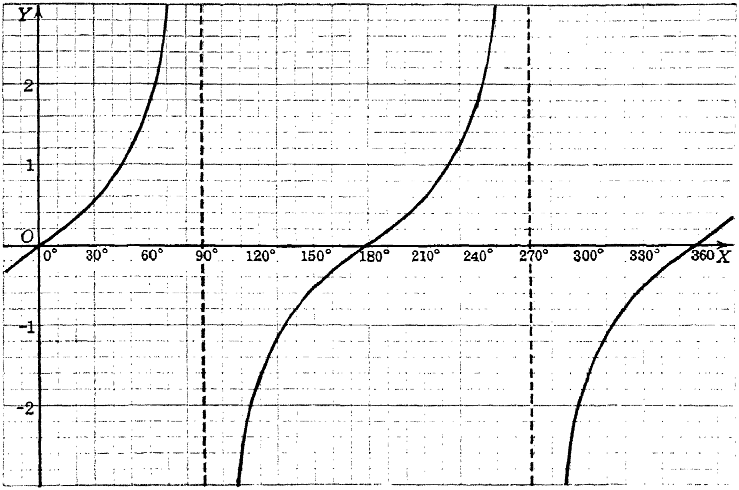
at 360°. The period of the cosine is also 360°. The graph of $y = \cos x$ is shown in Fig. 76.

87. Graphs of the tangent and the cotangent.

In Fig. 74 the value of the tangent is given by the length and the direction of the tangent line AT. Since this length is determined by the point of intersection of the tangent line at A with the terminal side of the angle, at 0° the tangent is 0. The tangent increases as the angle increases, until at 90° the terminal side is parallel to the tangent line, and there can be no point of intersection. That is, there is no value of the tangent for an angle of 90°. However, since the value of the tangent for an angle just less than 90° is

very great, and since the tangent is increasing as the angle increases, it is customary to say that the tangent approaches infinity (∞) as the angle approaches $90°$. (See section 38.)

In the second quadrant the terminal line must be prolonged backward to intersect the tangent line. This means that AT extends downward, and that the tangent is negative. As the angle increases beyond $90°$, the tangent, which



$y = \tan x$

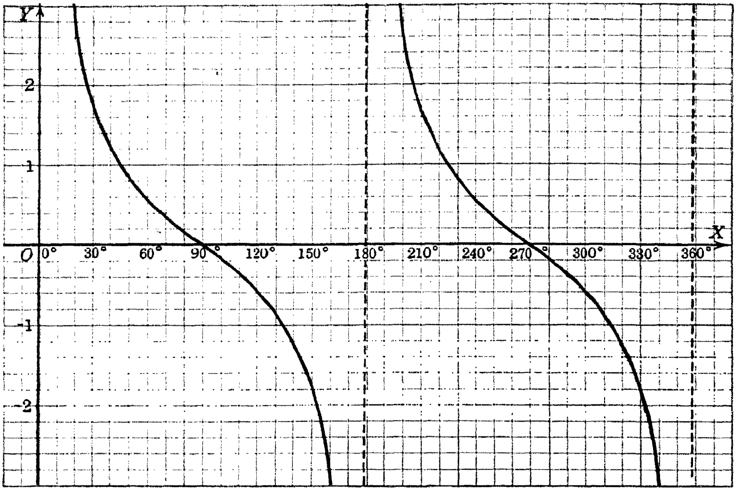
FIG. 77

has just extended indefinitely far in a positive direction, now begins at an indefinitely great distance in the negative direction.*

Thus, the tangent does not have a continuous change in value; there is a break at $90°$. It increases from very large negative values, for values of the angle just greater than

* When θ approaches $90°$ from below (i.e., in the first quadrant), the limit of $\tan \theta$ is $+\infty$; when θ approaches $90°$ from above (i.e., in the second quadrant), the limit of $\tan \theta$ is $-\infty$.

90°, to 0 at 180°. As the angle increases through the third quadrant, the terminal line must be prolonged backward, and the values are the same as in the first quadrant. As the angle increases from 270° to 360°, the tangent repeats



$y = \cot x$

FIG. 78

its values of the second quadrant. The tangent thus passes through a complete cycle of values twice in one complete rotation of the line generating the angle. Its period is consequently 180°.

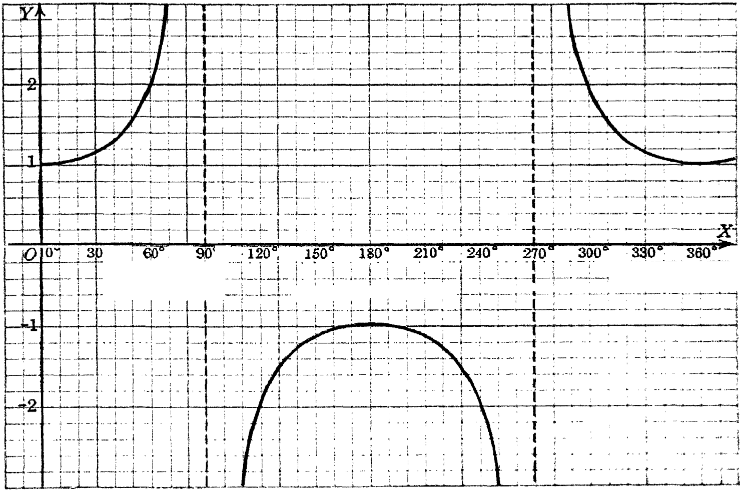
For a graph of $y = \tan x$ see Fig. 77.

In like manner, since the length and the direction of the cotangent line are determined by the intersection of the tangent line at B with the terminal side of the angle, the cotangent starts with very large values for very small positive values of the angle, and decreases to 0 at 90°. It continues to decrease through negative values in the second quadrant, these negative values becoming numerically greater and greater as the angle approaches 180°. As the

angle passes through 180°, the cotangent swings back to very large positive values, and decreases through 0 at 270° to very large negative values as the angle approaches 360°. (See Fig. 78.) Hence the cotangent also passes through a complete cycle of values twice in one complete rotation of the terminal line, and its period is 180°.

88. Graphs of the secant and the cosecant.

The secant starts with the value 1 at 0°, increases without bound as the angle approaches 90°, and jumps to very



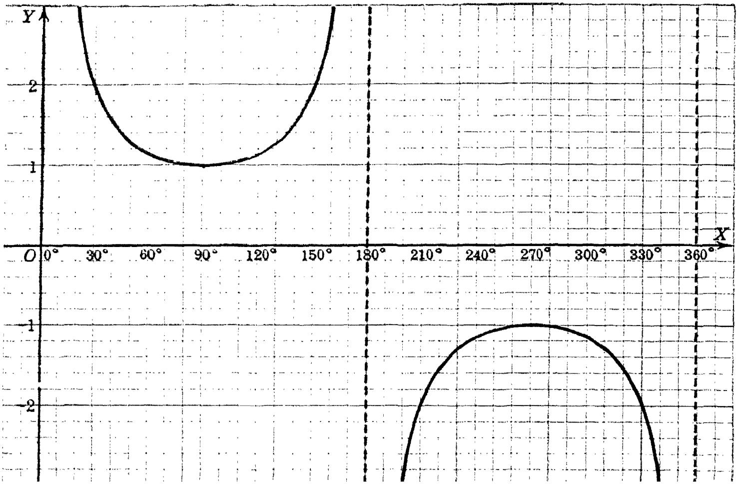
$y = \sec x$

FIG. 79

large negative values as the angle passes through 90°; it then increases to -1 at 180°, but decreases back through large negative values as the angle approaches 270°. As the angle passes through 270°, the secant changes sign and comes back to the value 1 at 360°. (See Fig. 79.) Its period is 360°.

The cosecant starts with very large values for small

values of the angle, decreases to 1 at 90°, and increases without bound as the angle approaches 180°. It then changes sign and rises from very large negative values to -1 as the angle increases to 270°, but recedes indefinitely



$y = \csc x$

FIG. 80

as the angle continues to 360°. (See Fig. 80.) Its period is 360°.

89. Use of radian measure in graphing.

It is sometimes desirable to use radian measure in constructing the graphs of the functions. In such cases the point on the x-axis which previously was marked 360° would be marked 2π radians, the point corresponding to 180° would be marked π, and so on. Here it is usual to take the same unit on each axis; thus, the point π would be 3.14+ units from the origin.

If the radian is used as the unit of measure of angle, the

period of sine, cosine, secant, and cosecant is 2π; the period of tangent and cotangent is π.

★90. Geometric construction of the sine and cosine graphs.

By using a unit circle, we can construct the sine curve as indicated in Fig. 81. In this figure a unit circle is drawn

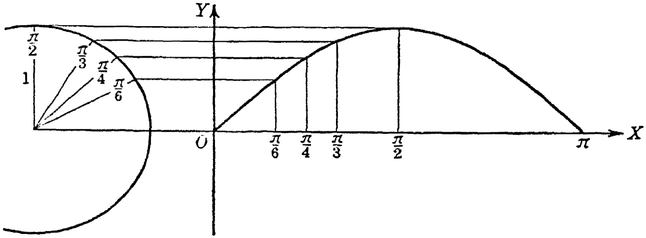


FIG. 81

at the left, and a horizontal line, to be used as the x-axis, is drawn through its center. On this line is marked an origin O, through which is drawn the y-axis. The segment from O to the point marked π is 3.1416 units long; that is, it is equal in length to the semicircumference. The distance from O to the point marked $\pi/6$ is equal to the arc of the circle from the point of its intersection with the x-axis to the point marked $\pi/6$, and so on. The method by which we obtain the ordinate corresponding to a given abscissa is evident from the figure.

The corresponding method of constructing the graph of the cosine curve is left as an exercise for the student.

EXERCISES X. A

Plot the following curves:

1. $y = 2 \sin x$. **2.** $y = 2 \cos x$. **3.** $y = \frac{1}{2} \sin x$.
4. $y = \sin 2x$. **5.** $y = \sin \frac{1}{2}x$. **6.** $y = \cos \frac{1}{2}x$.
7. $y = \cot \frac{1}{2}x$. **8.** $y = \sin 3x$. **9.** $y = \tan 2x$.
10. $y = \sin \pi x$. **11.** $y = \cos \frac{\pi x}{2}$. **12.** $y = \sin \frac{2\pi x}{3}$.

13. Plot $y = \sin x \cos x$.

SUGGESTION. $\sin x \cos x = \frac{1}{2} \sin 2x$.

14. In what points will a line one unit above the x-axis intersect the curve $y = \tan x$?

15. If the graphs of $y = \sin x$ and $y = \cos x$ are plotted on the same set of axes, for what values of x will they intersect?

16. Plot $y = \sin\left(x + \dfrac{\pi}{2}\right)$ and compare with $y = \cos x$.

17. Plot $y = \cos\left(\dfrac{\pi}{2} - x\right)$ and compare with $y = \sin x$.

18. Draw the graph of $y = \cos\left(x - \dfrac{\pi}{4}\right)$.

19. Draw the graph of $y = \sin(x - \frac{1}{2})$. Here radian measure is understood.

20. Given the equation $y = \sin x + \cos x$.

(a) Plot the curve by plotting the sine curve and the cosine curve separately and adding their ordinates geometrically (for example, by using dividers).

(b) Plot the curve by first reducing $\sin x + \cos x$ to the form $r \sin(x + \phi)$.

21. Draw the graph of $y = \sin x - \cos x$.

22. Plot $y = x + \sin x$, using radian measure.

23. Find the periods of the curves in exercises 1–12.

CHAPTER XI

Inverse Trigonometric Functions

91. Inverse trigonometric functions.

If $x = y^2$, then y is the positive or negative square root of x; in symbols, $y = \pm\sqrt{x}$. Similarly, if $x = \sin y$, then y is an angle whose sine is x; in abbreviated form we write

$$y = \arcsin x. \tag{1}$$

The right-hand member of this equation may be read " arc sine x " or " an angle whose sine is x," it being recalled that if a central angle of a unit circle is measured in radians, the intercepted arc is equal to the angle. The notation

$$y = \sin^{-1} x \tag{2}$$

is also used. The symbol $\sin^{-1} x$ may be read " inverse sine of x " or " antisine of x " or, to emphasize its meaning, " an angle whose sine is x." It should be carefully noted that the -1 is not an exponent. If we wish to have -1 as the exponent of a trigonometric function such as $\sin x$, we must write $(\sin x)^{-1}$, which means $1/\sin x$.

The function $\arcsin x$, or $\sin^{-1} x$, is called the **inverse sine function** of x. The other **inverse trigonometric functions** are

$$\begin{array}{lcl} \arccos x & \text{or} & \cos^{-1} x, \\ \arctan x & \text{or} & \tan^{-1} x, \\ \operatorname{arccot} x & \text{or} & \cot^{-1} x, \\ \operatorname{arcsec} x & \text{or} & \sec^{-1} x, \\ \operatorname{arccsc} x & \text{or} & \csc^{-1} x. \end{array}$$

92. Principal values.

An inverse trigonometric function, such as $\arcsin x$, has infinitely many values corresponding to each value of x. Consider, for example, $\arcsin \frac{1}{2}$. There are two angles less than 360° whose sine is $\frac{1}{2}$, namely 30° and 150°. Any angle obtained from either of these by adding or subtracting a multiple of 360° also has its sine equal to $\frac{1}{2}$. Therefore we may write

$$\arcsin \tfrac{1}{2} = 30° + n \cdot 360° \qquad \text{or} \qquad 150° + n \cdot 360° \qquad n = 0, \pm 1, \pm 2, \cdots \tag{1}$$

or, if we use radian measure, which is usually more desirable in dealing with the inverse functions,

$$\arcsin \tfrac{1}{2} = \frac{\pi}{6} + 2n\pi \qquad \text{or} \qquad \frac{5\pi}{6} + 2n\pi; \qquad n = 0, \pm 1, \pm 2, \tag{2}$$

The **principal value** of $\arcsin x$, which will be denoted by **Arcsin** x or $\mathbf{Sin^{-1}} x$, is that value between $-\pi/2$ and $\pi/2$ inclusive. Thus, the principal value of $\arcsin \frac{1}{2}$ is $-\pi/6$. If the principal value of $\arcsin x$ is θ, then all possible values are contained in the two sets

$$\theta + 2n\pi, \qquad \pi - \theta + 2n\pi; \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{3}$$

These two sets may be grouped together by the formula

$$n\pi + (-1)^n\theta; \qquad n = 0, \pm 1, \pm 2, \tag{4}$$

The notation for the principal values of the other inverse trigonometric functions is like that for the inverse sine, namely, **Arccos** x or $\mathbf{Cos^{-1}} x$, **Arctan** x or $\mathbf{Tan^{-1}} x$, etc.

The principal values of the inverse functions are defined as follows. That is, the principal value is that value in the interval specified.

$$
\begin{aligned}
-1 &\leqq x \leqq 1, & -\frac{\pi}{2} &\leqq \operatorname{Arcsin} x \leqq \frac{\pi}{2}, \\
-\infty &< x < \infty, & -\frac{\pi}{2} &< \operatorname{Arctan} x < \frac{\pi}{2}, \\
-1 &\leqq x \leqq 1, & 0 &\leqq \operatorname{Arccos} x \leqq \pi, \\
-\infty &< x < \infty, & 0 &< \operatorname{Arccot} x < \pi, \\
x &\geqq 1, & 0 &\leqq \operatorname{Arcsec} x < \frac{\pi}{2}, \\
x &\leqq -1, & -\pi &\leqq \operatorname{Arcsec} x < -\frac{\pi}{2}, \\
x &\geqq 1, & 0 &< \operatorname{Arccsc} x < \frac{\pi}{2} \\
x &\leqq -1, & -\pi &< \operatorname{Arccsc} x \leqq -\frac{\pi}{2}
\end{aligned}
$$

NOTE. Other definitions of the principal values of the inverse trigonometric functions for negative values of x are sometimes given. However, the foregoing definitions are the most convenient from the standpoint of calculus.

If the principal value of an inverse trigonometric function is θ, then all values of the inverse sine or of the inverse cosecant are given by (3) or (4). All values of the inverse cosine or of the inverse secant are given by

$$2n\pi \pm \theta; \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{5}$$

All values of the inverse tangent or of the inverse cotangent are given by

$$\theta + n\pi; \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{6}$$

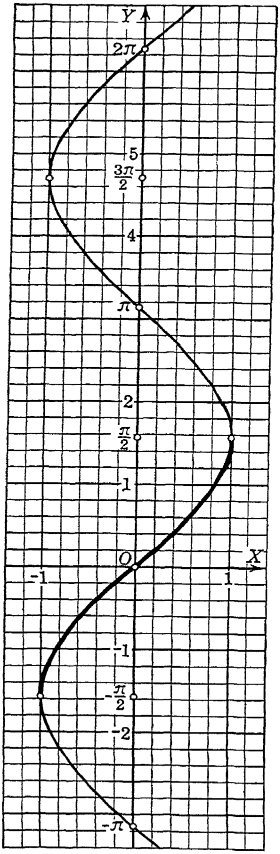
93. Graphs of the inverse trigonometric functions.

The graph of the equation

$$y = \arcsin x, \tag{7}$$

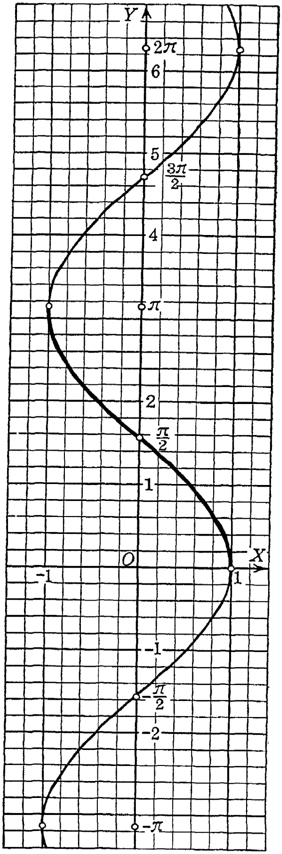
in which y is expressed in radians, is given in Fig. 82. The principal values of the function are indicated by the heavier

part of the curve, which constitutes the **principal branch** of the curve. It is clear that this curve is also the graph



$y = \arcsin x$

FIG. 82



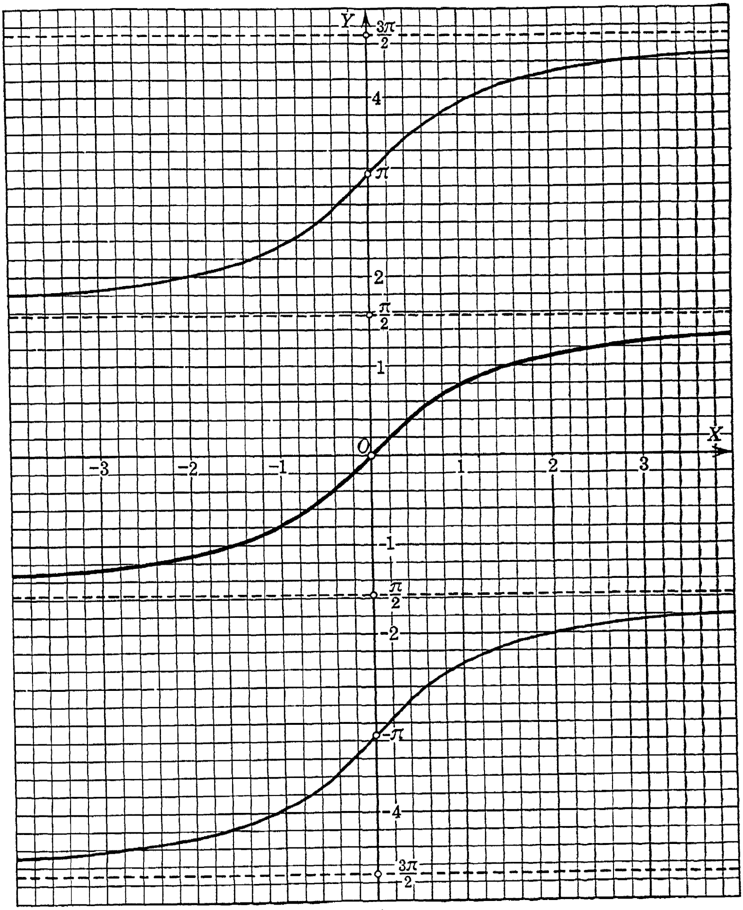
$y = \arccos x$

FIG. 83

of the equation $x = \sin y$, which is merely the other form of writing (7).

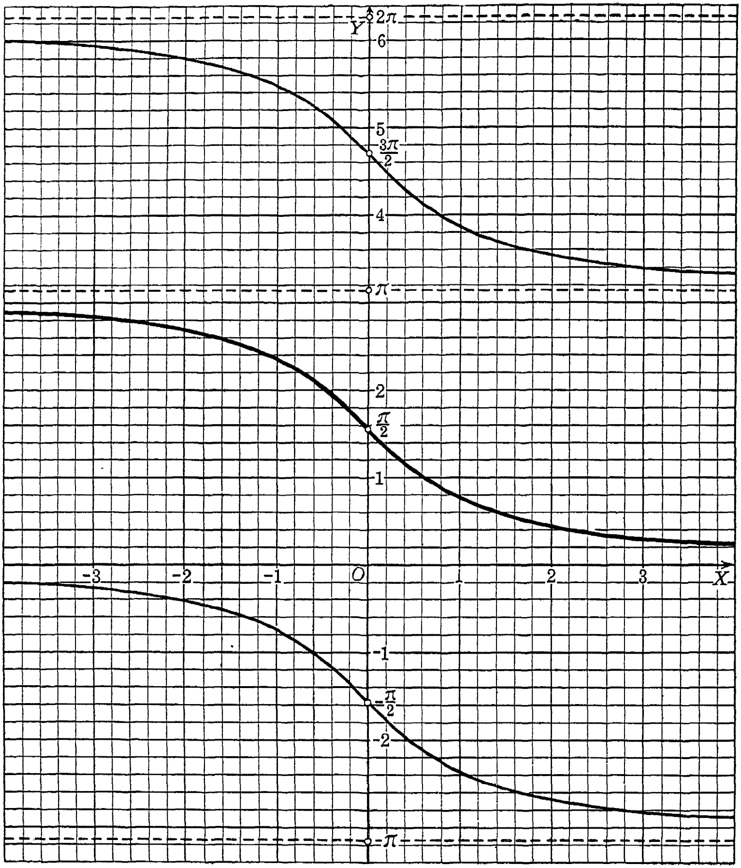
The graphs of the other inverse functions are shown in

Figs. 83–87. The principal branch in each case is indicated by the heavier part of the curve.



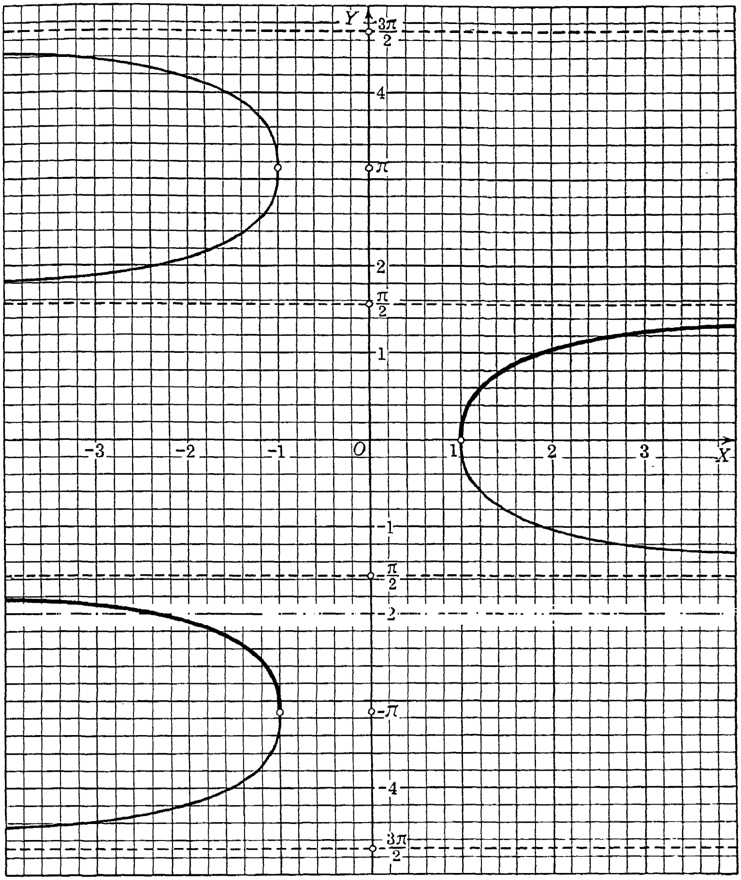
$y = \arctan x$

FIG. 84



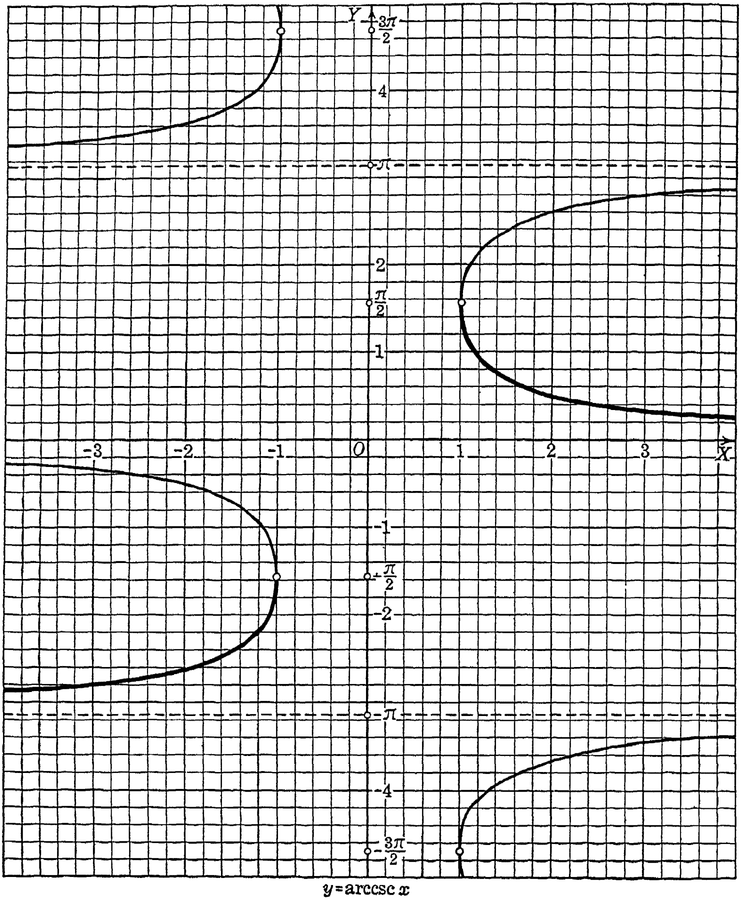
$y = \text{arccot}\, x$

FIG. 85



y arcsec x

FIG. 86



$y = \operatorname{arccsc} x$

FIG. 87

EXERCISES XI. A

1. Find $\arcsin \dfrac{\sqrt{3}}{2}$

SOLUTION. Let $\theta = \arcsin \dfrac{\sqrt{3}}{2} \cdot$ Then $\sin \theta = \dfrac{\sqrt{3}}{2}$, and the principal value of θ is $60°$ or $\pi/3$. Therefore, by (4),

$$\theta = n\pi + (-1)^n \frac{\pi}{3} \cdot$$

Find the principal values, and also the general values, of the following:

2. $\arcsin \frac{1}{2}$. **3.** $\operatorname{arccos}\left(-\dfrac{\sqrt{2}}{2}\right)$. **4.** $\arcsin 0$.

5. $\arccos 0$. **6.** $\operatorname{arccot} \dfrac{\sqrt{3}}{3} \cdot$ **7.** $\arctan 1$.

8. $\operatorname{arccsc}\sqrt{2}$. **9.** $\arctan(-\sqrt{3})$. **10.** $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)\cdot$

Find, by using tables, the principal values, and also the general values of

11. $\arcsin 0.23770$.
12. $\arccos 0.93590$.
13. $\arctan 1.4910$.
14. $\arcsin(-0.95510)$.
15. $\arccos(-0.01020)$.
16. $\arctan(-12.350)$.
17. $\arcsin \frac{2}{3}$.
18. $\arccos \frac{1}{8}$.
19. $\arctan 2$.
20. Find $\cos(\arctan \frac{5}{3})$.

SOLUTION. Let $\theta = \arctan \frac{5}{3}$. Then (see Fig. 88),

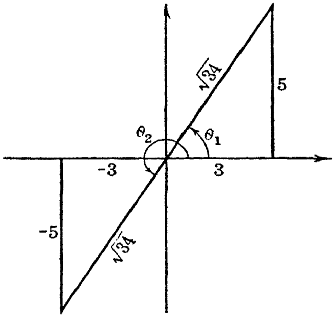


FIG. 88

$$\tan \theta = \frac{5}{3}, \qquad \cos \theta = \frac{3}{\pm\sqrt{34}} = \pm\frac{3\sqrt{34}}{34} \cdot$$

Find

21. $\tan(\text{Arcsin } \frac{3}{5})$.
22. $\sin(\text{Arccos } \frac{7}{25})$.
23. $\cos(\text{arccos } \frac{9}{13})$.
24. $\sin[\text{Arccos}(-\frac{15}{17})]$.
25. $\tan[\text{Arccos}(-\frac{15}{17})]$.
26. $\cot[\text{Arcsin}(-\frac{12}{13})]$.
27. $\sin(\arctan \frac{20}{21})$.
28. $\cos(\arcsin \frac{24}{25})$.
29. $\tan[\arccos(-\frac{4}{5})]$.
30. $\sec(\arctan 1.05)$.
31. $\cot[\arctan(-3)]$.
32. $\sec(\text{arccot } 2)$.
33. $\sin(\arcsin x)$.
34. $\cos(\arcsin x)$.
35. $\tan(\arcsin x)$.
36. $\sin(\arccos x)$.
37. $\cot(\arccos x)$.
38. $\tan(\arccos x)$.
39. $\sin(\arctan x)$.
40. $\cos(\arctan x)$.
41. $\sec(\arctan x)$.
42. $\tan(\text{arcsec } x)$.

43. Find the value of $\sin(\arccos \frac{3}{5} + \arctan \frac{12}{5})$.

Solution. Let $\theta = \arccos \frac{3}{5}$, $\phi = \arctan \frac{12}{5}$. Then,

$$\sin(\arccos \tfrac{3}{5} + \arctan \tfrac{12}{5}) = \sin(\theta + \phi)$$
$$= \sin\theta\cos\phi + \cos\theta\sin\phi = (\pm\tfrac{4}{5})(\pm\tfrac{5}{13}) + \tfrac{3}{5}(\pm\tfrac{12}{13}).$$

Using all possible combinations of signs, we find the following four distinct values for the above expression:

$$\tfrac{4}{13} + \tfrac{36}{65} = \qquad \tfrac{4}{13} - \tfrac{36}{65} = -\tfrac{16}{65},$$
$$-\tfrac{4}{13} + \tfrac{36}{65} = \qquad -\tfrac{4}{13} - \tfrac{36}{65} = -\tfrac{56}{65}.$$

They may be expressed in the more compact form: $\pm\frac{56}{65}$, $\pm\frac{16}{65}$.

Find the values of

44. $\sin(\text{Arcsin } \frac{7}{25} + \text{Arccos } \frac{4}{5})$.
45. $\cos(\text{Arcsin } \frac{15}{17} + \text{Arccot } \frac{9}{40})$.
46. $\tan(\arctan \frac{3}{4} + \arctan \frac{8}{15})$.
47. $\sin(\arcsin \frac{1}{3} + \arccos \frac{1}{3})$.
48. $\cos[\arcsin \frac{8}{17} + \arcsin(-\frac{3}{5})]$.
49. $\cos\left(2 \arcsin \frac{\sqrt{5}}{3}\right)$
50. $\sin(\frac{1}{2} \arccos \frac{7}{9})$.
51. $\tan(\arcsin \frac{5}{13} + 2 \arctan \frac{4}{5})$.
52. $\tan[\arctan \frac{3}{5} + \arcsin(-\frac{3}{5})]$.
53. $\sin(\arctan \frac{9}{40} - \text{arccot } \frac{21}{20})$.
54. $\cos[\text{arcsec } \frac{25}{7} - \arctan(-\frac{15}{8})]$.

55. $\sin(2 \arcsin \frac{3}{5} + \frac{1}{2} \arccos \frac{1}{49})$.

56. $\cos\left(\frac{1}{2} \arcsin \frac{\sqrt{15}}{} - 2 \arctan \frac{15}{8}\right)$.

57. $\sin(\arcsin \frac{4}{5} + \arctan \frac{12}{5} - \arccos \frac{8}{17})$.

58. Show that $\operatorname{Arctan} \dfrac{1}{2} + \operatorname{Arctan} \dfrac{1}{3} = \dfrac{\pi}{4}$.

Solution. Let $\theta = \operatorname{Arctan} \frac{1}{2}$, $\phi = \operatorname{Arctan} \frac{1}{3}$. Then we wish to prove that $\theta + \phi = \pi/4$.

$$\tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = 1.$$

From this we might have

$$\theta + \phi = \frac{\pi}{4} + n\pi; \qquad n = 0, \pm 1, \pm 2,$$

However, since we are dealing with principal values, θ and ϕ are in the interval from 0 to $\pi/2$. Therefore $\theta + \phi$ is in the interval from 0 to π, and we must have $\theta + \phi = \pi/4$.

Prove that

59. $\operatorname{Arcsin} \frac{3}{5} - \operatorname{Arctan} \frac{3}{5} = \operatorname{Arctan} \frac{3}{29}$.

60. $\operatorname{Arctan} \frac{1}{3} - \operatorname{Arctan} \frac{1}{4} = \operatorname{Arctan} \frac{1}{13}$.

61. $\operatorname{Arcsin} \frac{3}{5} + \operatorname{Arcsin} \frac{8}{17} = \operatorname{Arcsin} \frac{77}{85}$.

62. $\operatorname{Arccos} \frac{4}{5} + \operatorname{Arccos} \frac{12}{13} = \operatorname{Arccos} \frac{33}{65}$.

63. $\operatorname{Arccos} \frac{4}{5} + \operatorname{Arctan} \frac{3}{5} = \operatorname{Arctan} \frac{27}{11}$.

64. $2 \operatorname{Arctan} \dfrac{1}{3} + \operatorname{Arctan} \dfrac{1}{7} = \dfrac{\pi}{4}$.

65. $\operatorname{Arccos} \frac{63}{65} + 2 \operatorname{Arctan} \frac{1}{5} = \operatorname{Arcsin} \frac{3}{5}$.

66. $\operatorname{Arctan} \frac{1}{4} + \operatorname{Arctan} \frac{2}{9} = \frac{1}{2} \operatorname{Arccos} \frac{3}{5}$.

67. $\operatorname{Arctan} \dfrac{1}{2} + \operatorname{Arctan} \dfrac{1}{5} + \operatorname{Arctan} \dfrac{1}{8} = \dfrac{\pi}{4}$.

68. Prove that $\operatorname{Arctan} x + \operatorname{Arctan} y = \operatorname{Arctan} \dfrac{x + y}{1 - xy}$, provided the value of the left-hand side is between $-\pi/2$ and $\pi/2$.

Note. In general,

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy},$$

if it is understood that the particular values assigned to two of the inverse functions are arbitrary; the particular value of the third is determined when the values of the others are assigned.

Prove that

69. $\text{Arcsin}\, x + \text{Arccos}\, x = \frac{\pi}{2}$ for $-1 \leqq x \leqq 1$.

70. $\text{Arctan}\, x + \text{Arccot}\, x = \frac{\pi}{2}$ for all values of x.

71. $2\, \text{Arcsin}\, x = \text{Arccos}(1 - 2x^2)$ for $0 \leqq x \leqq 1$.

72. $\text{Arcsin}\, x = \pm \text{Arccos}\, \sqrt{1 - x^2}$, according as $x \gtrless 0$.

73. $\text{Arctan}\, x = \text{Arcsin}$; for all values of x.

74. $\text{Arctan}\, \frac{2x}{1 - x^2} = \text{Arcsin}\, \frac{}{1 + x^2}$ for $-1 < x < 1$.

75. $\text{Arctan}\, x + \text{Arccot}(x + 1) = \text{Arctan}(x^2 + x + 1)$ for all values of x.

76. Find all possible values of $\arcsin(\cos\theta)$.

SOLUTION. Let $\phi = \arcsin(\cos\theta)$. Then,

$$\sin\phi = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right).$$

Therefore,

$$\phi = \begin{cases} \frac{\pi}{2} - \theta + n \cdot 2\pi, \\ \pi - \left(\frac{\pi}{2} - \theta\right) + n \cdot 2\pi. \end{cases}$$

These two sets of solutions may be expressed in the form

$$\phi = \frac{\pi}{2} \pm \theta + 2n\pi.$$

Find all possible values of the following expressions:

77. $\arcsin(\sin\theta)$.

78. $\arccos(\cos\theta)$.

79. $\arctan(\tan\theta)$.

80. $\arccos(\sin\theta)$.

CHAPTER XII

Trigonometric Equations

94. Trigonometric equations.

An equation which is satisfied by certain values only of the unknown quantity or quantities that it contains is called a **conditional equation.** Examples of conditional equations are $2x - 1 = 0$, which is satisfied by $x = \frac{1}{2}$ only; $x^2 + y^2 = 25$, which is satisfied by an infinite number of pairs of values of x and y, but certainly not by all pairs of values; $\sin\theta = \frac{1}{2}$, which is satisfied by $\theta = 30°$, $150°$, $390°$, $510°$, etc., but not by all values of θ.

An **identical equation,** or **identity,** is an equation which is satisfied by all values (with perhaps some exceptions *) of the unknown quantity or quantities which it contains. Examples of identities are

$$\begin{aligned}(x+1)^2 &= x^2 + 2x + 1,\\ \sin^2\theta + \cos^2\theta &= 1,\\ \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi.\end{aligned}$$

The equations † which we shall consider in this chapter are conditional equations, identities having already been considered in various places throughout the book.

Trigonometric equations require, for a complete solution, general expressions such as (1) or (2) in section 92 of the preceding chapter. However, the equation is sometimes

* For example, the identity $\tan\theta = \sin\theta/\cos\theta$ is not defined for values of θ, such as $\pi/2$, which make the denominator of the right-hand side equal to zero.

† It is customary to omit the qualifying adjective, and to refer to a conditional equation merely as an "equation."

considered sufficiently solved if all positive values of the unknown quantity less than 360° are obtained, or if the principal value of an inverse function is obtained.

There is no general method of solving trigonometric equations. If the equation contains a single function of an angle, solve for this function by appropriate algebraic methods, and then find the corresponding values of the angle. If more than one function appears in the equation, the equation should ordinarily be transformed so that it contains only one function, or into a factored form so that each factor contains only one function.

When the equation involves functions of different angles, such as θ, 2θ, $\frac{1}{2}\theta$, $\theta + 45°$, it can sometimes be reduced to an equivalent equation which contains but a single function of a single angle, or to an equivalent equation which can be separated into factors each of which contains a single function of a single angle.

As in algebra, the test for each solution of an equation is to substitute it in the original equation to determine whether it satisfies the equation.

Some of the methods of solving trigonometric equations will be illustrated by examples.

Example 1.

Solve the equation $\sin\theta = \cos\theta$.

SOLUTION. Divide both sides by $\cos\theta$:*

$$\tan\theta = 1.$$

The principal value of θ is 45°. The two positive values of θ less than 360° are 45° and 225°. The complete solution is

$$= 45° + n \cdot 180°, \quad \text{or} \quad = \frac{\pi}{4} + n\pi; \quad n = 0, \pm 1, \pm 2,$$

* When both sides of an equation are divided by a quantity containing the unknown, this quantity should be set equal to zero to obtain possible solutions. If we set $\cos\theta = 0$, we get $\theta = 90°, 270°, \cdots$. However, these values are not solutions of the equation $\sin\theta = \cos\theta$.

This equation can also be solved by replacing $\cos\theta$ by $\pm\sqrt{1-\sin^2\theta}$ and squaring both sides:

$$\begin{aligned} \sin^2\theta &= 1 - \sin^2\theta, \\ 2\sin^2\theta &= 1, \\ \sin\theta &= \pm\frac{1}{\sqrt{2}}, \\ \theta &= 45°, 135°, 225°, 315°, \cdots. \end{aligned}$$

If this method is used, all the values obtained must be tested. It will be found that 135° and 315° do not satisfy the original equation. They are **extraneous solutions** introduced by squaring, and must be discarded.

Example 2.

Solve: $$\cos^2\theta + 2\sin\theta + 1 = 0.$$

SOLUTION. Replacing $\cos^2\theta$ by $1 - \sin^2\theta$, we get, after a slight simplification,

$$\sin^2\theta - 2\sin\theta - 2 = 0.$$

This is a quadratic equation in $\sin\theta$; solving it by the quadratic formula, we find

$$= 1 \pm 1.73205 = 2.73205 \text{ or } -0.73205.$$

The first value must be discarded, since the sine cannot be greater than 1; from the second we get two values of θ between 0° and 360°, viz.,

$$\begin{aligned} \theta &= 180° + 47°\,3.5' = 227°\ \ 3.5', \\ \theta &= 360° - 47°\,3.5' = 312°\,56.5'. \end{aligned}$$

The general solution is given by

$$\theta = \begin{cases} 227°\ \ 3.5' + n\cdot 360°, \\ 312°\,56.5' + n\cdot 360°; \end{cases} \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Example 3.

Solve: $$2 \sin^2 \theta - \cos 2\theta = 0.$$

SOLUTION. Replace $\cos 2\theta$ by $1 - 2 \sin^2 \theta$, and combine like terms:

$$\begin{aligned} 4 \sin^2 \theta - 1 &= 0, \\ \sin \theta &= \pm\tfrac{1}{2}, \\ \theta &= 30°, 150°, 210°, 330°, \cdots. \end{aligned}$$

The general solution may be written in the form

$$\theta = n \cdot 180° \pm 30° = n\pi \pm \frac{\pi}{6} \cdot$$

Equations of the form $a \cos \theta \pm b \sin \theta = c$ can be solved by reducing the left side to one of the forms $r \sin(\theta \pm \phi)$, $r \cos(\theta \pm \phi)$. (See section 76.)

Example 4.

Solve: $$3 \sin \theta - 4 \cos \theta = 1.$$

SOLUTION. Divide both sides by $\sqrt{3^2 + (-4)^2} = 5$:

$$\tfrac{3}{5} \sin \theta - \tfrac{4}{5} \cos \theta = \tfrac{1}{5} = 0.2. \quad (1)$$

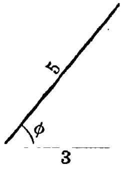


FIG. 89

If ϕ is an angle such that (see Fig. 89)

$$\cos \phi = \tfrac{3}{5}, \qquad \sin \phi = \tfrac{4}{5}, \quad (2)$$

then (3) takes the form

$$\sin \theta \cos \phi - \cos \theta \sin \phi = 0.2,$$

or

$$\sin(\theta - \phi) = 0.2.$$

But from (2), using tables, we find $\phi = 53° \, 8'$. Therefore

$$\begin{aligned} \sin(\theta - 53° \, 8') &= 0.2, \\ \theta - 53° \, 8' &= 11° \, 32', 168° \, 28', \cdots, \\ \theta &= 64° \, 40', 221° \, 36', \cdots. \end{aligned}$$

This method of solution is particularly valuable if the numbers involved are not simple, since it is adapted to the use of logarithms.

The equation could be solved by making the substitution $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ and following the method used for solving radical equations in algebra. (Cf. example 1, second method.) This, however, introduces extraneous solutions.

EXERCISES XII. A

Solve the following equations:

1. $2\cos^2\theta - \sin^2\theta = 2.$

2. $2\cos^2\theta + 3\sin\theta = 0.$

3. $\tan\theta + \cot\theta = 2.$

4. $\sin\theta = 2\cos\theta.$

5. $\sec\theta = 4\csc\theta.$

6. $\cos 2\theta + \sin\theta = 0.$

7. $\sin 2\theta + \cos\theta = 0.$

8. $\sin 2\theta = 3\sin^2\theta - \cos^2\theta.$

9. $\sin^2\theta = 1 - \sin 2\theta.$

10. $\tan^2\theta = \sin 2\theta.$

11. $\sin 2\theta + 2\cos 2\theta = 1.$

12. $4\sec^2 2\theta + \tan 2\theta = 7.$

13. $\sin 2\theta = \cos 3\theta.$

Solution. $\sin 2\theta = \cos 3\theta = \sin(90° - 3\theta).$

Now if $\sin\theta = \sin\phi$, it follows that either

$$\theta = \phi + n \cdot 360° \qquad \text{or} \qquad \theta = 180° - \phi + n \cdot 360°.$$

In the present case, therefore,

$$2\theta = 90° - 3\theta + n \cdot 360° \qquad \text{or} \qquad 2\theta = 180° - (90° - 3\theta) + n \cdot 360°.$$

The first equation yields

$$5\theta = 90° + n \cdot 360°, \qquad \theta = 18° + n \cdot 72°.$$

The second can be reduced to $\theta = 270° + n \cdot 360°.$

14. $\sin\theta = \cos(\theta + 15°).$

15. $\sin(\theta + 10°) = \cos(\theta - 40°).$

16. $\sin(15° - 2\theta) = \cos(7\theta + 10°).$

17. $\tan 5\theta = \cot 3\theta.$

18. $\tan(\theta + 25°) = \cot 2\theta.$

19. $\tan(2\theta - 18°) = \cot(3\theta + 48°).$

20. $\cos\theta + \cos 2\theta + \cos 3\theta = 0.$

21. $\csc 2\theta + \cot 2\theta = 2.$

22. $\sin 2\theta \cos 2\theta = -2 \sin \theta$.
23. $\sin \theta + \cos \theta = 1$.
24. $5 \cos \theta + 12 \sin \theta = 4$.
25. $3264 \sin \theta - 5728 \cos \theta = 6018$.
26. $0.1723 \cos \theta + 1.3284 \sin \theta = 0.8492$.
27. $\sqrt{3} \cos \theta - \sin \theta = \sqrt{2}$.
28. $\csc \theta = \cot \theta + \sqrt{3}$.
29. $2 \sin^2 \theta + \sin^2 2\theta = 2$.
30. $\tan^2 \theta + \cot^2 \theta = \frac{10}{3}$.
31. $\cos 3\theta - 2 \cos 2\theta + \cos \theta = 0$.
32. $\sin(\theta + 12°) + \sin(\theta - 8°) = \sin 20°$.
33. $\sin^4 \theta - \cos^4 \theta = \frac{7}{36}$.
34. $\sin^4 \theta + \cos^4 \theta = 1$.
35. $\sin 3\theta = \cos 2\theta - 1$.
36. $3 - 4 \cos^2 \theta = \cos 3\theta$.
37. $\sin(60° - \theta) - \sin(60° + \theta) = \dfrac{\sqrt{3}}{2}$.
38. $\tan(\theta + 15°) = 3 \tan(\theta - 15°)$.
39. Solve the following simultaneous equations for r and θ in terms of x and y:

$$x = r \cos \theta,$$
$$y = r \sin \theta.$$

40. Solve the following simultaneous equations for r, θ, ϕ in terms of x, y, z, restricting r to positive values:

$$x = r \sin \theta \cos \phi,$$
$$y = r \sin \theta \sin \phi,$$
$$z = r \cos \theta.$$

Solve for θ and ϕ:

41. $\sin \theta - \sin \phi = 0.7038$, $\cos \theta - \cos \phi = -0.7245$.
42. $\cos \theta + \cos \phi + \frac{1}{2} = 0$, $\cos \frac{1}{2}\theta + \frac{1}{2} \cos \phi - \frac{1}{4} = 0$.
43. $\sin \theta = \tan \phi$, $\cos \theta \cos \phi = \frac{1}{2}$.
44. $\sin \theta + \sin \phi = a$, $\cos \theta + \cos \phi = b$.
45. Solve the equation $\cos x = x$ (x in radians).

SOLUTION. Draw the graphs of $y = \cos x$ and $y = x$. (See Fig. 90.) The value of x for which the curve and the line intersect is the solution of the equation. According to the graph, this value is approximately $x = 0.74$, about 42° 24′.

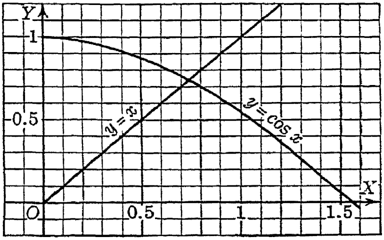


FIG. 90

A more accurate value may be obtained by writing the equation in the form $\cos x - x = 0$, and employing interpolation. Tabulating for several values of x, we get the results shown below.

	x	$\cos x$	$\cos x - x$
42° 20′	.73886	.73924	.00038
42° 21′	.73915	.73904	−.00011
42° 22′	.73944	.73885	−.00059
42° 23′	.73973	.73865	−.00108
42° 24′	.74002	.73846	−.00156

Since we want the value of $\cos x - x$ to be zero, the required value of x is between 0.73886 and 0.73915. Using the ordinary methods of interpolation, we have

$$\frac{0.73886}{0.73915 - 0.73886} = \frac{0 - 0.00038}{-0.00011 - 0.00038},$$

or

$$\frac{x - 0.73886}{0.00029} \quad \frac{38}{49};$$

from which we get

$$x = 0.73886 + \tfrac{38}{49} \times 0.00029$$

$$= 0.73886 + 0.00022 = 0.73908.$$

By means of more extensive tables, the value correct to five decimal places is found to be 0.73909.

Solve the following equations, in which x is to be expressed in radians:

46. $\cos x = 2x.$

47. $\sin x = x - 1.$

48. $\sin x = \frac{1}{x}\cdot$

49. $\tan x = 1 - x.$

50. $\sin x = \log_{10} x.$

51. $\cos x = x^2.$

52. $\log_{10} x + x = 0.$

53. $x = 2 + \pi \sin x.$

54. $x = 1 + \frac{\pi}{6} \sin x.$

55. $x = \sin 2x.$

56. $3^x = 2 \cos x.$

57. $\sin x = 10^x.$

58. A horizontal cylindrical tank is 10 feet long and 4 feet in diameter. It contains 10 gallons of liquid. How deep is the liquid? (1 gal. = 231 cu. in.)

Some of the following equations are conditional, some are identical. Solve the former, prove the latter.

59. $\frac{\sin^2 \theta}{1 + \cos \theta} = 1 - \cos$

60. $\frac{\sin^2 \theta}{1 + \sin \theta} = 1 - \sin$

61. $\cos 2\theta + \sin 2\theta = (\cos \theta + \sin \theta)^2 + 2 \sin^2 \theta.$

62. $\cos 2\theta + \sin 2\theta = (\cos \theta + \sin \theta)^2 - 2 \sin^2 \theta.$

63. $\cot \frac{1}{2}\theta = \cot \theta(1 + \sec \theta).$

64. $\csc 2\theta + 2 \tan \theta = 3.$

65. $2 \csc 2\theta - \tan \theta = \cot \theta.$

CHAPTER XIII

★ Complex Numbers

95. Imaginary and complex numbers.

The **imaginary unit,** denoted by i, is a number having the property $i^2 = -1$. We postulate that it obeys all the laws of addition and multiplication assumed for real numbers.

Since $i^3 = i^2 \cdot i = -i, i^4 = (i^2)^2 = 1, i^5 = i^4 \cdot i = i, \cdots$, it is seen that the successive integral powers of i run through the cycle $i, -1, -i, 1$.

A number of the form $a + bi$, in which a and b are real numbers, is called a **complex number.** The number a is called the **real part,** and bi is called the **imaginary part** of the complex number, b being the coefficient of the imaginary part. If $b \neq 0$, the complex number is called an **imaginary number.** If $b \neq 0$ and $a = 0$, the complex number reduces to the form bi, which is called a **pure imaginary number.** If both a and b are different from zero, the number is sometimes called a **mixed imaginary number.** If $b = 0$, the complex number reduces to the real number a.

Two complex numbers such as $a + bi$ and $a - bi$, which differ only in the signs of their imaginary parts, are called **conjugate** complex numbers. Either is said to be the conjugate of the other.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. In particular, $a + bi = 0$ if and only if $a = 0$ and $b = 0$.

96. Operations with complex numbers.

By definition, addition or subtraction of complex numbers is effected by adding or subtracting their real parts to

obtain the real part of their sum or difference, and by adding or subtracting their imaginary parts to obtain the imaginary part of their sum or difference. Thus,

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \tag{1}$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i. \tag{2}$$

We multiply complex numbers according to the laws of real numbers, simplifying results by making use of the relation $i^2 = -1$. Thus,

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i. \end{aligned} \tag{3}$$

Division of complex numbers can be accomplished by writing the quotient in fractional form and multiplying both numerator and denominator by the conjugate of the denominator. Thus, to divide $a + bi$ by $c + di$ (c and d not both zero) we write

$$\begin{aligned}\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} &= \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}. \end{aligned} \tag{4}$$

97. Geometric representation of complex numbers.

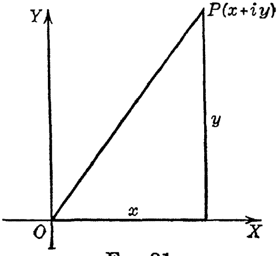


FIG. 91

The complex number $x + yi$ may be represented by the point whose abscissa is x and whose ordinate is y. (See Fig. 91.) When complex numbers are so represented, the horizontal axis is called the **axis of real numbers,** and the vertical axis is called the **axis of imaginary numbers.**

98. Geometric addition and subtraction of complex numbers.

Let the complex numbers $a + bi$ and $c + di$ be represented by the points M and N respectively, and their sum, $(a + c) + (b + d)i$, by the point P. (See Fig. 92.) Draw OM, ON, MP, NP. Drop NQ, MR, PS perpendicular to OX. Draw MT parallel to OX. Then,

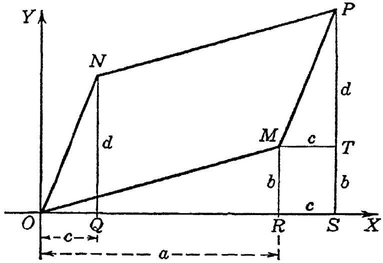


FIG. 92

$$MT = RS = OS - OR = (a + c) - a = c = OQ,$$
$$TP = SP - ST = (b + d) - b = d = QN.$$

Also, angle TPM is equal to angle QNO, and MP is parallel to ON. Quadrilateral $OMPN$ is a parallelogram, since two of its sides are both equal and parallel.

Thus, *to add two complex numbers geometrically, complete the parallelogram which has as adjacent sides the lines drawn from the origin to the points representing the two numbers. The fourth vertex of the parallelogram will be the point representing the sum of the two numbers.*

If we think of the complex numbers $a + bi$ and $c + di$ as represented by the vectors OM and ON in Fig. 92, the sum of the numbers will be the vector OP. (See section 67.)

To subtract $c + di$ from $a + bi$ geometrically, we may add $a + bi$ and $-c - di$.

EXERCISES XIII. A

Perform the indicated operations geometrically:

1. $(2 + 5i) + (6 + i)$. **2.** $(3 + 4i) + (5 - 2i)$.
3. $(5 + 3i) - (3 - 2i)$. **4.** $(-4 + 2i) + (3 + 5i)$.
5. $(3i) + (6 + 2i)$. **6.** $(5i) + (6)$.

7. $(5) - (6 - 7i)$.
8. $(1 + 2i) + (3 + 6i)$.
9. $(-6 + i) + (7 + 2i)$.
10. $(3 + 6i) - (1 + 2i)$.
11. $(7 + 5i) + (7 - 5i)$.
12. $(7 + 5i) - (7 - 5i)$.
13. $(-5 - 5i) + (10 + 3i)$.
14. $(8 + 6i) - (4 + 6i)$.
15. $(-3 + 2i) + (3 - 7i)$.
16. $(5 + 7i) + (5 + 7i)$.
17. $(10 + 2i) + (-2 + 5i) + (3 - 4i)$.

Suggestion. Combine the first two numbers graphically, and then combine their sum with the third.

18. $(5 + 6i) + (6 - 2i) - (4 - 7i)$.
19. Given the complex numbers $10 - 4i$, $5 + 5i$, $1 - 6i$. Show that the same result is obtained by geometrically (a) adding the first and second and then adding their sum to the third, (b) adding the first and third and then adding their sum to the second, (c) adding the second and third and then adding their sum to the first.

99. Trigonometric form of complex numbers.

Let the complex number $x + yi$ be represented by the point P in Fig. 93. As usual, let $OP = r$ (a non-negative number), and denote the angle XOP by θ. Then,

$$x = r \cos \theta, \qquad y = r \sin \theta, \tag{1}$$

and the complex number may be written

$$r(\cos \theta + i \sin \theta), \tag{2}$$

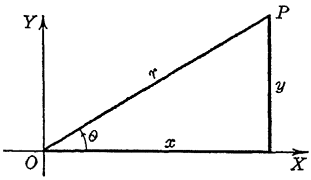


Fig. 93

which is called the **trigonometric** or **polar form** of the complex number, the form $x + yi$ being called the **rectangular form.** The expression $\cos \theta + i \sin \theta$ is sometimes abbreviated **cis θ.**

In the trigonometric form (2), r is called the **modulus** or the **absolute value** of the complex number, θ is called the **amplitude** or the **argument.** We have

$$r = \qquad\qquad \tan \theta = \frac{y}{x}\cdot \tag{3}$$

Example 1.

Reduce $3 + 4i$ to trigonometric form.

SOLUTION.

$$r = \sqrt{3^2 + 4^2} = 5,$$

$$\tan \theta = \tfrac{4}{3} = 1.3333, \qquad \theta = 53.1°,$$

$$3 + 4i = 5(\cos 53.1° + i \sin 53.1°).$$

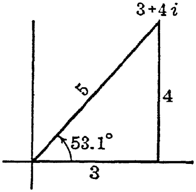


FIG. 94

Example 2.

Reduce $-1 + i\sqrt{3}$ to trigonometric form.

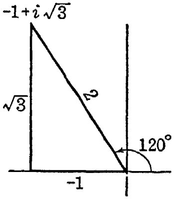


FIG. 95

SOLUTION.

$$r = \sqrt{1 + 3} = 2,$$

$$\tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}, \qquad \theta = 120°,$$

$$-1 + i\sqrt{3} = 2(\cos 120° + i \sin 120°).$$

EXERCISES XIII. B

Reduce to trigonometric form:

1. $-5 + 5i$. **2.** $3 + 4i$. **3.** $\sqrt{3} + i$.
4. $6 + 6i$. **5.** $3 - 4i$. **6.** $5 + 5i\sqrt{3}$.
7. $6i$. **8.** -10. **9.** $-8 - 15i$.
10. $12 - 5i$. **11.** $2 + 3i$. **12.** $12 + 5i$.
13. $5 - i$. **14.** $-5i$. **15.** $-7 - 7i$.
16. $6 - 6i$. **17.** $6 - 8i$. **18.** $-2\sqrt{3} + 2i$.
19. $-7 + 2i$. **20.** $10 - 8i$. **21.** $\frac{1}{2} + \frac{1}{3}i$.

Reduce to rectangular form:

22. $2(\cos 60° + i \sin 60°)$. **23.** $5(\cos 45° + i \sin 45°)$.
24. $7(\cos 30° + i \sin 30°)$. **25.** $3(\cos 225° + i \sin 225°)$.
26. $4(\cos 330° + i \sin 330°)$. **27.** $10(\cos 90° + i \sin 90°)$.
28. $5(\cos 180° + i \sin 180°)$. **29.** $4(\cos 270° + i \sin 270°)$.
30. $8(\cos 150° + i \sin 150°)$. **31.** $\sqrt{2}(\cos 315° + i \sin 315°)$.
32. $\sqrt{3}(\cos 210° + i \sin 210°)$. **33.** $10[\cos(-35°) + i \sin(-35°)]$.
34. $8(\cos 100° + i \sin 100°)$. **35.** $5(\cos 200° + i \sin 200°)$.
36. $2(\cos 300° + i \sin 300°)$. **37.** $10(\cos 400° + i \sin 400°)$.

100. Multiplication and division of complex numbers in trigonometric form.

A very interesting result is obtained if two complex numbers expressed in trigonometric form are multiplied together. Thus,

$$\begin{aligned} &r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &\quad = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &\quad = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \qquad (1) \end{aligned}$$

Therefore, *the product of two complex numbers is a complex number whose modulus is the product of the moduli of the numbers, and whose amplitude is the sum of their amplitudes.*

It can readily be seen that this holds for the product of any finite number of complex numbers.

If one complex number is divided by another,* we have

$$\begin{aligned} \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\ &= \frac{r_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &\ \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]. \qquad (2) \end{aligned}$$

In words, *the quotient of two complex numbers is a complex number whose modulus is the modulus of the dividend divided by the modulus of the divisor, and whose amplitude is the amplitude of the dividend minus the amplitude of the divisor.*

EXERCISES XIII. C

Perform the indicated operations, first reducing the numbers to trigonometric form (if necessary):

1. $3(\cos 40° + i\sin 40°)\cdot 5(\cos 70° + i\sin 70°)$.
2. $2(\cos 200° + i\sin 200°)\cdot 6(\cos 300° + i\sin 300°)$.

* The divisor cannot be zero.

3. $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)(2 + 2i)$. 4. $(-3 + 3i)(3 - i\sqrt{3})$.
5. $6(\cos 70° + i \sin 70°) \cdot 2(\cos 40° + i \sin 40°)$.
6. $10(\cos 20° + i \sin 20°) \cdot 5(\cos 70° + i \sin 70°)$.
7. $(3 + 3i\sqrt{3}) \div (\sqrt{3} - i)$. 8. $(-5 + 5i\sqrt{3}) \div (3 + 3i)$.
9. $(6 - 6i) \div (-2 + 2i\sqrt{3})$. 10. $(1 + i) \div (1 + i\sqrt{3})$.

101. Powers of complex numbers.

Raising to a power is a special case of multiplication, and it follows, by a repeated application of (1) of section 100, that

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta),$$

where n is a positive integer. The foregoing relation is known as **De Moivre's theorem.***

Example.

Find the value of $(1 + i)^5$.

SOLUTION. Plot the complex number $1 + i$ (Fig. 96). The absolute value is $\sqrt{2}$ and the amplitude is 45°.

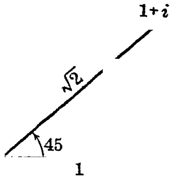


FIG. 96

$$(1 + i)^5 = [\sqrt{2}(\cos 45° + i \sin 45°)]^5$$
$$= 4\sqrt{2}(\cos 5 \cdot 45° + i \sin 5 \cdot 45°)$$
$$= 4\sqrt{2}(\cos 225° + i \sin 225°) = -4(1 + i).$$

102. Roots of complex numbers.

To prove De Moivre's theorem for the case in which the exponent is the reciprocal of a positive integer, take the expression

$$[r(\cos \theta + i \sin \theta)]^{1/n} = r^{1/n}(\cos \theta + i \sin \theta)^{1/n}. \qquad (1)$$

* A formal proof of the theorem can be effected by the process of mathematical induction. For an explanation of this process, see the author's *College Algebra*, Chapter X.

Let $\theta = n\phi$. Then the right side of (1) reduces to

$$r^{1/n}(\cos n\phi + i \sin n\phi)^{1/n} = r^{1/n}[(\cos \phi + i \sin \phi)^n]^{1/n} = r^{1/n}(\cos \phi + i \sin \phi),$$

or

$$[r(\cos \theta + i \sin \theta)]^{1/n} = r^{1/n}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right). \qquad (2)$$

Since for any whole number k,

$$\cos(\theta + k \cdot 360°) = \cos \theta, \qquad \sin(\theta + k \cdot 360°) = \sin \theta,$$

we have

$$[r(\cos \theta + i \sin \theta)]^{1/n} = [r\{\cos(\theta + k \cdot 360°) + i \sin(\theta + k \cdot 360°)\}]^{1/n}$$
$$= r^{1/n}\left(\cos \frac{\theta + k \cdot 360°}{n} + i \sin \frac{\theta + k \cdot 360°}{n}\right). \qquad (3)$$

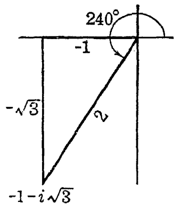


FIG. 97

By giving values to k from 0 to $n - 1$ inclusive, we obtain n distinct roots of the number $r(\cos \theta + i \sin \theta)$.

Example.

Find the fourth roots of $-1 - i\sqrt{3}$.

SOLUTION. Plot the number $-1 - i\sqrt{3}$ (Fig. 97) and note that

$$-1 - i\sqrt{3} = 2(\cos 240° + i \sin 240°),$$

$$(-1 - i\sqrt{3})^{\frac{1}{4}} = 2^{\frac{1}{4}}\left(\cos \frac{240° + k \cdot 360°}{4} + i \sin \frac{240° + k \cdot 360°}{4}\right)$$
$$= \sqrt[4]{2} \cos(60° + k \cdot 90°) + i \sin(60° + k \cdot 90°).$$

Giving k successively the values 0, 1, 2, 3, we find for the four distinct fourth roots of $-1 - i\sqrt{3}$:

$$\sqrt[4]{2}(\cos 60° + i \sin 60°)$$
$$= \sqrt[4]{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\sqrt[4]{2} + \frac{i}{2}\sqrt[4]{18},$$

$\sqrt[4]{2}(\cos 150° + i \sin 150°)$

$$= \sqrt[4]{2}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -\frac{1}{2}\sqrt[4]{18} + \frac{i}{2}\sqrt[4]{2},$$

$\sqrt[4]{2}(\cos 240° + i \sin 240°)$

$$= \sqrt[4]{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\sqrt[4]{2} - \frac{i}{2}\sqrt[4]{18},$$

$\sqrt[4]{2}(\cos 330° + i \sin 330°)$

$$= \sqrt[4]{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{1}{2}\sqrt[4]{18} - \frac{i}{2}\sqrt[4]{2}.$$

In Fig. 98, P represents the complex number $2(\cos 240° + i \sin 240°)$; P_1, P_2, P_3, P_4 represent the four roots whose amplitudes are 60°, 150°, 240°, 330°, respectively.

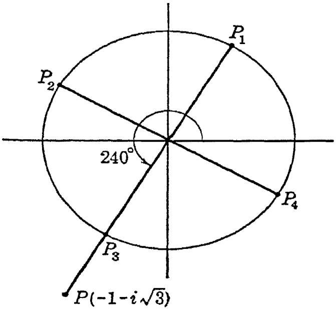


FIG. 98

Note that the roots can be found geometrically as follows: Draw a circle with center at the origin and with radius equal to the numerical fourth root of the absolute value of the number whose fourth roots are to be found, that is, a radius equal to $\sqrt[4]{2}$. Take one-fourth of the amplitude of the original number ($\frac{1}{4} \times 240° = 60°$). This locates the point P_1 on the circle. The four roots all lie on the circle and are spaced at equal intervals of 90°. Thus we can find P_2, P_3, P_4.

In general, the nth roots of the complex number $r(\cos \theta + i \sin \theta)$ can be found as follows: Draw a circle whose center is the origin and whose radius is the numerical nth root of r; divide the angle θ by n, the index of the root. Now divide the circumference of the circle, from θ/n to $\theta/n + 360°$, into n equal parts. The n points of division will be the required roots.

EXERCISES XIII. D

Use De Moivre's theorem to raise to the indicated powers:

1. $[7(\cos 18° + i \sin 18°)]^3$. **2.** $[\sqrt{3}(\cos 20° + i \sin 20°)]^8$.

3. $(1 + i)^{10}$. **4.** $(\sqrt{3} + i)^7$.

5. $(5 - 5i)^4$.

6. $[\sqrt{2}(\cos 100° + i \sin 100°)]^{10}$

7. $(\cos 22° + i \sin 22°)^8$.

8. $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^7$.

9. $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3$.

10. $[2(\cos 10° + i \sin 10°)]^{-3}$.

11. $[10(\cos 70° + i \sin 70°)]^{-6}$

12. $(1 + i)^{-10}$.

Find all of the

13. Square roots of $9(\cos 80° + i \sin 80°)$.

14. Square roots of $4(\cos 100° + i \sin 100°)$.

15. Cube roots of $27(\cos 27° + i \sin 27°)$.

16. Square roots of $1 + i\sqrt{3}$.

17. Cube roots of $1 + i\sqrt{3}$.

18. Cube roots of $-\sqrt{3} + i$.

19. Cube roots of 1.

SUGGESTION. $1 = \cos 0° + i \sin 0°$.

20. Fifth roots of -1.

21. Sixth roots of $-8i$.

22. Cube roots of $-2 + 3i$.

23. Fifth roots of $-4 - 4i$.

24. Seventh roots of $\sqrt{2}(1 - i)$.

Obtain all of the roots of the following equations:

25. $x^5 - 1 = 0$.

26. $x^3 + 1 = 0$.

27. $x^4 + 1 = 0$.

28. $x^5 + 32 = 0$.

29. $x^4 - 16i = 0$.

30. $x^7 - 1 = 0$.

31. $x^4 + x^3 + x^2 + x + 1 = 0$.

SUGGESTION. Multiply by $x - 1$, solve the resulting equation, and discard the extraneous root $x = 1$.

32. $x^4 - x^3 + x^2 - x + 1 = 0$.

SPHERICAL TRIGONOMETRY

CHAPTER XIV

Introduction to Spherical Trigonometry

103. Definitions and propositions from solid geometry.

The intersection of a plane with a sphere is a circle. If the plane passes through the center of the sphere, the intersection is a **great circle**; otherwise the intersection is a **small circle.** Obviously the radius of a great circle is equal to the radius of the sphere, while the radius of a small circle is less than the radius of the sphere.

A line through the center of the sphere perpendicular to the plane of a circle is called the **axis** of the circle. This axis pierces the sphere in two points, which are called the **poles** of the circle.

The shortest distance in space between two points on a sphere is the straight line joining them, but this line does not lie on the surface of the sphere. The shortest path on the sphere between the two points is the arc (not greater than a semicircle) of a great circle joining the points. The **distance** (on the sphere) between the two points is defined to be the length of this arc. This distance is usually expressed in angular units, and is equal to the angle which the arc subtends at the center of the sphere. However, it can be converted into linear units if the radius of the sphere is known.

104. Spherical triangles.

A **spherical triangle** is that part of the surface of a sphere bounded by three arcs of great circles.* Like a plane tri-

* That part of the surface of a sphere bounded by the arcs of two great circles is called a **lune.**

angle, it is composed of six parts—three sides and three angles. We shall ordinarily designate the angles by A, B, C, and the opposite sides by a, b, c, respectively.

To each spherical triangle there corresponds a trihedral angle whose vertex is at the center of the sphere. A spherical triangle, with the corresponding trihedral angle, is illustrated in Fig. 99. In this figure, O is the center of the sphere. The sides of the spherical triangle are measured by the corresponding face angles of the trihedral angle. Thus, a is measured by BOC, b is measured by AOC, c is measured by AOB.

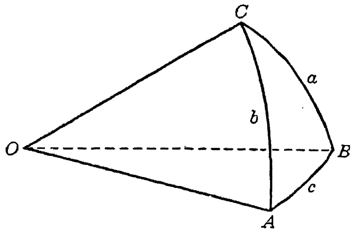


FIG. 99

The angles of the spherical triangle are measured by the corresponding dihedral angles of the trihedral angle. For example, angle A is measured by the dihedral angle whose edge is OA, namely $B\text{–}OA\text{–}C$.

This follows if the angle A of the spherical triangle is defined as the angle between the tangents at A to the arcs AB and AC, since the angle between these tangents is the plane angle of the dihedral angle.

It is possible to have spherical triangles with one or more sides or angles greater than 180°. However, we shall consider only triangles for which each side and each angle is less than 180°.* For such triangles, the sum of the sides is less than 360°, and the sum of the angles is between 180° and 540°; that is,

$$a + b + c < 360°, \qquad (1)$$

$$180° < A + B + C < 540°. \qquad (2)$$

* Note that even with this restriction it is possible to have a spherical triangle with two, or even three, right angles. A spherical triangle having a right angle is called a **right spherical triangle,** one with two right angles is said to be **birectangular,** while one with three right angles is called **trirectangular.**

The amount by which the sum of the angles of a spherical triangle exceeds 180° is called the **spherical excess** of the triangle. That is, if E denotes the spherical excess, then

$$E = A + B + C - 180° \tag{3}$$

The sum of any two sides is greater than the third side, and their difference is less than the third side.

If two sides are equal, the angles opposite are equal.

If two angles are equal, the sides opposite are equal.

If two sides are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.

If two angles are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.

105. Spherical polygons.

A **spherical polygon** is that part of the surface of a sphere bounded by three or more arcs of great circles. To every spherical polygon there corresponds a polyhedral angle whose vertex is at the center of the sphere. The sides of the polygon are measured by the corresponding face angles of the polyhedral angle, and the angles of the polygon are measured by the corresponding dihedral angles of the polyhedral angle.

A spherical polygon of n sides can be divided into $n - 2$ triangles by drawing diagonals from one vertex. The sum of the excesses of these triangles is equal to the sum of the angles of the polygon less $(n - 2) \cdot 180°$. This difference may be called the **spherical excess** of the polygon.

106. Polar triangles.

With the vertices of a spherical triangle ABC as poles, construct three great circles. The great circles whose poles are B and C will intersect in two diametrically opposite

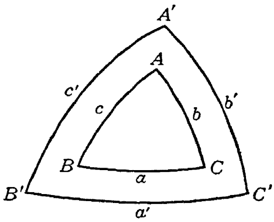


FIG. 100

points. Denote by A' that point of intersection which is on the same side of BC as is A. Determine B' and C' similarly. Then $A'B'C'$ is the **polar triangle** of ABC. (See Fig. 100.) Conversely, ABC is the polar triangle of $A'B'C'$.

Each angle of a spherical triangle is the supplement of the corresponding side in the polar triangle. That is,

$$A + a' = 180^\circ, \qquad B + b' = 180^\circ, \qquad C + c' = 180^\circ,$$
$$A' + a = 180^\circ, \qquad B' + b = 180^\circ, \qquad C' + c = 180^\circ.$$

107. Areas.

The area of the surface of a sphere of radius R is $4\pi R^2$.

The area of a spherical triangle on a given sphere is proportional to its spherical excess. Since the area of a trirectangular triangle (whose excess is $270^\circ - 180^\circ = 90^\circ$) is one-eighth of the surface of the sphere, that is, $\frac{1}{8} \cdot 4\pi R^2 = \frac{1}{2}\pi R^2$, we have for the area of a triangle ABC,

$$\frac{\text{area}}{\frac{1}{2}\pi R^2} = \frac{E}{90},$$

or,

$$\text{area} = \frac{\pi R^2 E}{180} \tag{1}$$

This formula applies to any spherical polygon provided the excess of the polygon is defined as in section 105.

A **spherical degree** is a unit of surface measurement on a sphere equal to half a lune whose angle is 1°. (For definition of lune see footnote, page 197.) The area, in spherical degrees, of a spherical triangle, or of any spherical polygon, is equal to its spherical excess.*

* When the three sides of a spherical triangle are known, the excess can be determined by **L'Huilier's formula,** given here without derivation:

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)},$$

in which $s = \frac{1}{2}(a + b + c)$.

CHAPTER XV

Solution of Right Spherical Triangles

108. Formulas for solving right spherical triangles.

In Fig. 101 is represented a right spherical triangle, ABC, with the right angle at C (this will be the usual notation) and with sides a and b each less than 90°. Also shown is the

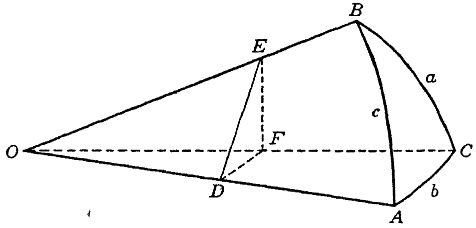


Fig. 101

trihedral angle $O\text{–}ABC$ associated with the triangle, O being the center of the sphere.

Through any point E on the edge OB, pass a plane DEF perpendicular to the edge OA and intersecting this edge in D. Then DE and DF will be perpendicular to OA.

From the various constructions it follows that the plane triangles ODF, ODE, OFE, DFE are right triangles, the vertex of the right angle being named as the middle letter.

In triangle DFE, angle D is equal to angle A of the spherical triangle, and each of the other plane right triangles has an angle equal to one of the sides of the spherical triangle.

Making use of these facts, we have

$$\sin a = \sin FOE = \frac{FE}{OE}, \qquad \sin c = \sin DOE = \frac{DE}{OE},$$

$$\frac{\sin a}{\sin c} \quad \frac{FE}{DE} = \sin A. \tag{1}$$

Also,

$$\tan b = \tan DOF = \frac{DF}{OD}, \qquad \tan c = \tan DOE = \frac{DE}{OD},$$

$$\frac{\tan b}{\tan c} = \frac{DF}{DE} = \cos A. \tag{2}$$

Similarly,

$$\tan a = \tan FOE = \frac{FE}{OF}, \qquad \sin b = \sin DOF = \frac{DF}{OF},$$

$$\frac{\tan a}{\sin b} = \frac{FE}{DF} = \tan A. \tag{3}$$

Finally,

$$\cos a = \cos FOE = \frac{OF}{OE}, \qquad \cos b = \cos DOF = \frac{OD}{OF},$$

$$\cos a \cos b = \frac{OD}{OE} \quad \cos c. \tag{4}$$

If the plane DEF had been constructed perpendicular to OB instead of to OA, we should have been led to results similar to (1), (2), (3), which can be obtained from these formulas by interchanging A and B, a and b. They are

$$\frac{\sin b}{\sin c} = \sin B, \qquad \frac{\tan a}{\tan c} = \cos B, \qquad \frac{\tan b}{\sin a} = \tan B. \tag{5}$$

Note that when this interchange is applied to (4) the formula reverts into itself.

From the foregoing formulas it can further be proved that

$$\cos a \sin B = \cos A, \qquad \cos b \sin A = \cos B, \tag{6}$$

$$\cot A \cot B = \cos a \cos b = \cos c. \tag{7}$$

Collecting these numbered results, and clearing of frac-

tions when necessary, we have the following ten formulas for the solution of right spherical triangles:

$$\sin a = \sin c \sin A, \quad (8) \qquad \sin b = \sin c \sin B, \quad (9)$$
$$\tan a = \sin b \tan A, \quad (10) \qquad \tan b = \sin a \tan B, \quad (11)$$
$$\tan a = \tan c \cos B, \quad (12) \qquad \tan b = \tan c \cos A, \quad (13)$$
$$\cos c = \cos a \cos b, \quad (14) \qquad \cos c = \cot A \cot B, \quad (15)$$
$$\cos A = \cos a \sin B, \quad (16) \qquad \cos B = \cos b \sin A. \quad (17)$$

They have been derived for the case in which each part of the spherical triangle ABC (except the right angle C) is less than 90°. However, it can be proved that they hold for parts equal to or greater than 90°.

109. Napier's rules.

The foregoing ten formulas may, by a clever device due to Napier, be put into a form which is easily remembered.

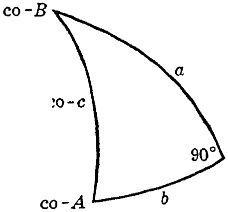


Fig. 102

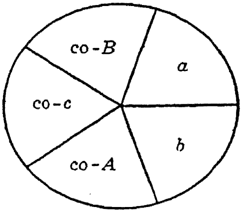


Fig. 103

In the schematic triangle of Fig. 102 we have replaced A by the symbol co-A, meaning "complement of A," and similarly for B and c.* Note that angle C is omitted. The five parts may also be arranged in a circle, as in Fig. 103, and are consequently often referred to as **circular parts.**

If in either of these diagrams any part is called the **middle part,** the two parts next to it are called the **adjacent parts,** and the other two are called the **opposite parts.** For example, if a is the middle part, then b and co-B are the adjacent parts, co-c and co-A are the opposite parts. **Napier's rules** are:

* It should be understood that Fig. 102 does not represent a triangle.

I. The sine of any middle part is equal to the product of the tangents of the adjacent parts.

II. The sine of any middle part is equal to the product of the cosines of the opposite parts.

As an illustration, let us take the part a. Rule I gives

$$\sin a = \tan b \tan \text{co-}B = \tan b \cot B,$$

which is formula (11). Rule II gives

$$\sin a = \cos \text{co-}c \cos \text{co-}A = \sin c \sin A,$$

which is (8).

By applying Napier's rules to each of the five parts of the diagram of Fig. 102 or that of Fig. 103, we obtain all ten of the formulas (8) to (17).

As a further mnemonic scheme we observe that the vowel **i** occurs in "sine" and "m**i**ddle," the vowel **a** predominates in "t**a**ngents" and "**a**dj**a**cent" of Rule I, while the vowel **o** predominates in "c**o**sines" and "**o**pp**o**site" of Rule II.

110. Solution of right spherical triangles.

If any two parts of a right spherical triangle (in addition to the right angle C) are given, the remaining parts can be found. However, it should be noted that sometimes no solution exists. (See example 2 later in this section.)

The quadrant in which any required part terminates may be determined by noting the signs of the functions involved. However, if the unknown part is determined from its sine, there are two possibilities for this part, the tabular value and its supplement, and consequently there are two solutions, subject however to the restrictions of the following theorems:

THEOREM I. *In a right spherical triangle, any side and the opposite angle terminate in the same quadrant.*

From equation (16), namely

$$\cos A = \cos a \sin B,$$

it is seen, since $\sin B$ is positive, that $\cos a$ and $\cos A$ must

have the same sign. That is, a and A terminate in the same quadrant. The same result can be proved for b and B.

THEOREM II. *If any two of the three parts a, b, c, terminate in the same quadrant, the third terminates in the first quadrant; if any two terminate in different quadrants, the third terminates in the second quadrant.*

The proof follows directly from equation (14),

$$\cos c = \cos a \cos b.$$

For if any two of the functions $\cos a$, $\cos b$, $\cos c$ have like signs, the third is positive; if any two have unlike signs, the third is negative.

The solution of a right spherical triangle can always be checked by the formula involving the three computed parts.

Example 1.

In a right spherical triangle ($C = 90°$), $A = 69° \ 50.8'$, $c = 72° \ 15.4'$; find B, a, b.

SOLUTION.	A	$69° \ 50.8'$
	c	$72° \ 15.4'$
$\sin a = \sin c \sin A$,	$\log \sin c$	$9.97884 - 10$
$\log \sin a = \log \sin c + \log \sin A$.	$\log \sin A$	$9.97256 - 10$
	$\log \sin a$	$9.95140 - 10$
		63° 23.8′ *
$\cos A = \tan b \cot c$,	$\log \cos A$	$9.53723 - 10$
$\log \tan b = \log \cos A - \log \cot c$.	$\log \cot c$	$9.50511 - 10$
	$\log \tan b$	0.03212
	b	**47° 7.0′**
$\cos c = \cot A \cot B$,	$\log \cos c$	$9.48395 - 10$
$\log \cot B = \log \cos c - \log \cot A$.	$\log \cot A$	$9.56467 - 10$
	$\log \cot B$	$9.91928 - 10$
	B	**50° 17.7′**
CHECK.† $\sin a = \tan b \cot B$,	$\log \tan b$	0.03212
$\log \sin a = \log \tan b + \log \cot B$.	$\log \cot B$	$9.91928 - 10$
	$\log \sin a$	$9.95140 - 10$

* The supplementary value is not admissible, since, by Theorem I, a and A must terminate in the same quadrant.

† This check verifies the consistency of the logarithms, but does not prove that the angular quantities are correct.

Example 2.

Solve the spherical triangle $C = 90°$, $A = 120°$, $a = 100°$.

SOLUTION.	A	120°	
		100°	
$\sin b = \tan a \cot A$,	$\log \tan a$	0.75368	(neg)
$\log \sin b = \log \tan a + \log \cot A$.	$\log \cot A$	9.76144 − 10	(neg)
	$\log \sin b$	0.51512	
No solution.		impossible	

Example 3.

Given $C = 90°$, $B = 36° \; 42.2'$, $b = 30° \; 17.5'$; find the remaining parts.

SOLUTION.	B	36° 42.2′
		30° 17.5′
$\sin a = \tan b \cot B$,	$\log \tan b$	9.76654 − 10
$\log \sin a = \log \tan b + \log \cot B$.	$\log \cot B$	0.12757
	$\log \sin a$	9.89411 − 10
		51° 35.6′ or **128° 24.4′**
$\sin b = \sin c \sin B$,	$\log \sin b$	9.70278 − 10
$\log \sin c = \log \sin b - \log \sin B$.	$\log \sin B$	9.77646 − 10
	$\log \sin c$	9.92632 − 10
		57° 33.6′ or **122° 26.4′**
$\cos B = \cos b \sin A$,	$\log \cos B$	9.90403 − 10
$\log \sin A$	$\log \cos b$	9.93624 − 10
$= \log \cos B - \log \cos b$.	$\log \sin A$	9.96779 − 10
		68° 12.2′ or **111° 47.8′**
CHECK. $\sin a = \sin c \sin A$,	$\log \sin c$	9.92632 − 10
$\log \sin a = \log \sin c + \log \sin A$.	$\log \sin A$	9.96779 − 10
	$\log \sin a$	9.89411 − 10

By Theorems I and II, the obtained values are grouped into the following two solutions:

$$A = 68° \; 12.2', \qquad a = 51° \; 35.6', \qquad c = 57° \; 33.6';$$
$$A' = 111° \; 47.8', \qquad a' = 128° \; 24.4', \qquad c' = 122° \; 26.4'.$$

* The notation (neg) indicates that the function is negative.

EXERCISES XV. A

Find the remaining parts of the following triangles, in each of which $C = 90°$:

1. $A = 80° 10.5'$, $c = 110° 46.3'$.
2. $B = 130° 30.0'$, $a = 114° 23.8'$.
3. $B = 36° 42.5'$, $c = 112° 25.0'$.
4. $A = 136° 5.2'$, $a = 110° 18.6'$.
5. $A = 75° 15.0'$, $B = 133° 8.0'$.
6. $a = 66° 59.5'$, $b = 156° 34.3'$.
7. $B = 154° 44.3'$, $b = 156° 3.0'$.
8. $A = 116° 32.4'$, $b = 50° 25.6'$.
9. $B = 112° 19.7'$, $a = 77° 35.3'$.
10. $a = 39° 46.3'$, $b = 62° 30.6'$.
11. $a = 130° 12.9'$, $c = 73° 58.0'$.
12. $A = 19° 15.3'$, $B = 85° 33.0'$.
13. $b = 26° 28.7'$, $c = 61° 25.1'$.
14. $A = 132° 15.6'$, $B = 47° 44.4'$.
15. $a = 98° 8.1'$, $c = 77° 41.9'$.
16. $B = 124° 14.8'$, $b = 147° 15.2'$.
17. $A = 25° 16.6'$, $a = 18° 54.3'$.
18. $A = 69° 2.4'$, $a = 62° 12.8'$.
19. $A = 75° 21.9'$, $b = 14° 59.6'$.
20. $B = 83° 56.7'$, $b = 77° 21.8'$.
21. Three concurrent edges of a cube are OP, OQ, OR. Find the dihedral angle between the plane PQR and one of the faces of the cube.
22. Show that if $B = C = 90°$, then $b = c = 90°$, and that A and a are indeterminate, but $A = a$.
23. Show that if $c = C = 90°$, then either $A = a = 90°$, and B and b are indeterminate, but $B = b$; or else $B = b = 90°$, and A and a are indeterminate, but $A = a$.
24. Show that if C is a right angle and if $b = c$ (and consequently each is a right angle), then $B = 90°$, and that A and a are indeterminate, but $A = a$.

111. Quadrantal triangles.

A **quadrantal triangle** is a spherical triangle having a side equal to $90°$. The polar triangle of a quadrantal triangle is

a right triangle, which can be solved by the methods explained in the preceding section. The parts of the quadrantal triangle can then be obtained.

For example, suppose we have given $c = 90°$, $b = 50°$, $A = 70°$. We know that

$$C' = 180° - c = 90°, \qquad B' = 180° - b = 130°,$$
$$a' = 180° - A = 110°.$$

We then find A', b', c', from which the values of a, B, C are readily obtained.

EXERCISES XV. B

Solve the following quadrantal triangles ($c = 90°$):

1. $a = 70°\ 7.8'$, $b = 52°\ 36.7'$.
2. $C = 135°\ 33.7'$, $a = 31°\ 30.7'$.
3. $A = 118°\ 46.4'$, $C = 100°\ 7.8'$.
4. $B = 55°\ 47.1'$, $C = 105°\ 9.5'$.
5. $A = 102°\ 38.3'$, $a = 96°\ 3.3'$.
6. $A = 73°\ 45.4'$, $b = 123°\ 36.1'$.
7. $a = 106°\ 38.6'$, $b = 36°\ 49.7'$.
8. $A = 122°\ 39.7'$, $a = 116°\ 52.5'$.
9. $B = 63°\ 4.6'$, $b = 69°\ 29.7'$.
10. $a = 60°\ 39.8'$, $b = 65°\ 52.4'$.

112. Isosceles spherical triangles.

The great circle drawn from the vertex of an isosceles spherical triangle to the midpoint of the opposite side divides the triangle into two symmetric right triangles. The solution of an isosceles spherical triangle can thus be reduced to the solution of a right spherical triangle.

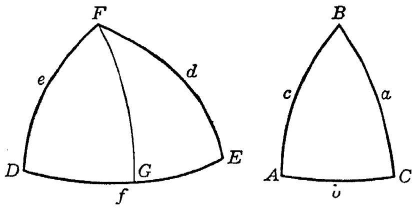


FIG. 104

Example.

Find the remaining parts of an isosceles spherical triangle in which the equal angles are $D = E = 80°\ 27'$ and the side included by these equal angles is $f = 76°\ 42'$. (See Fig. 104.)

SOLUTION. Draw a perpendicular, FG, from the vertex F to the base DE. This divides the triangle into two symmetric right spherical triangles DFG and GFE. For clarity, the first of these has been redrawn at the right in Fig. 104, and has been relettered, so that A, B, C replace D, F, G, respectively. Then, in the triangle ABC, we have $C = 90°$, $b = \frac{1}{2}f$. The logarithmic work follows.

$\cos B = \cos b \sin A,$

$\log \cos B = \log \cos b + \log \sin A.$

	A	$80° \; 27'$
	b	$38° \; 21'$
log cos	b	$9.89445 - 10$
log sin	A	$9.99394 - 10$
log cos	B	$9.88839 - 10$
	B	$39° \; 20.5'$

$\cos A = \tan b \cot c,$

$\log \cot c = \log \cos A - \log \tan b.$

log cos	A	$9.21987 - 10$
log tan	b	$9.89827 - 10$
log cot	c	$9.32160 - 10$
	c	$78° \; 9'$

Returning to the isosceles triangle, we have

$$F = 2B = 2 \times 39° \; 20.5' = 78° \; 41',$$
$$d = e = c = 78° \; 9'.$$

EXERCISES XV. C

Solve the following triangles:

1. $A = C = 69° \; 2.3'$, $b = 93° \; 16.4'$.
2. $B = C = 52° \; 36.7'$, $b = 73° \; 58.0'$.
3. $B = 112° \; 47.8'$, $a = c = 99° \; 9.6'$.
4. $a = c = 77° \; 7.7'$, $b = 37° \; 30.4'$.
5. $A = 153° \; 48.2'$, $a = 145° \; 3.8'$, $B = C$.
6. $A = C = 77° \; 40.5'$, $b = 52° \; 1.8'$.
7. $A = B = 95° \; 5.1'$, $C = 100° \; 10.8'$.
8. $A = 58° \; 58.8'$, $b = c = 63° \; 47.8'$.
9. $A = 62° \; 1.5'$, $a = c = 71° \; 59.3'$.
10. $B = 72° \; 48.8'$, $b = 64° \; 50.6'$, $a = c$.
11. $a = b = c = 10°$.
12. $a = b = c = 80°$.
13. $a = b = c = 100°$.
14. $A = B = C = 80°$.
15. $A = B = C = 100°$.
16. $A = B = C = 170°$.

17. Show that if each side of a spherical triangle is 60° each angle is arccos $\frac{1}{3}$.

18. Show that if each angle of a spherical triangle is 120° each side is arccos $(-\frac{1}{3})$.

19. Show that if each side of a spherical triangle is 30° each angle is arccos $(2\sqrt{3} - 3)$.

20. Prove that in an equilateral spherical triangle

$$\cos A = \frac{\cos a}{1 + \cos a}$$

21. Prove that in an equiangular spherical triangle

$$\cos a = \frac{\cos A}{1 - \cos A}$$

22. In an isosceles spherical triangle the base is 63° 8.8′ and the equal sides are 40° 4.4′. Find the perpendicular from the vertex to the base, also the perpendicular from one end of the base to the opposite side.

CHAPTER XVI

Solution of Oblique Spherical Triangles

113. Oblique spherical triangles.

If no angle of a spherical triangle is a right angle the triangle is **oblique**. For the solution of oblique spherical triangles, certain formulas, analogous to those of Chapter VII are needed, and we shall proceed to develop them.

114. Law of sines.

Let ABC be any spherical triangle. Through the vertex C draw the arc of a great circle perpendicular to the

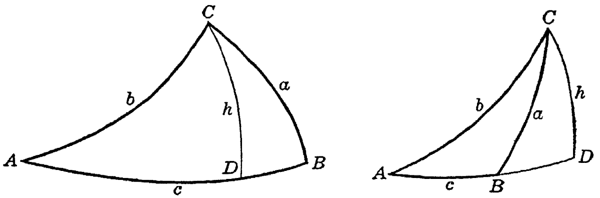


FIG. 105

side c (produced if necessary) at the point D. (See Fig. 105.) Designate the length of this perpendicular CD by h.

The foregoing construction yields two right spherical triangles, ADC and BDC. By Napier's rules we find

$$\sin h = \sin a \sin B, \qquad \sin h = \sin b \sin A. \tag{1}$$

Equating the two values of $\sin h$, and dividing by $\sin A \sin B$, we get

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}. \tag{2}$$

Similarly, by drawing an arc through the vertex B perpendicular to the side b, we can prove the relation

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C} \tag{3}$$

Combining (2) and (3), we obtain the **law of sines** for spherical triangles,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{4}$$

That is, *the sines of the sides of a spherical triangle and the sines of the corresponding opposite angles are in proportion.*

115. Law of cosines for sides.

In Fig. 106, in which the construction is the same as that in Fig. 105, denote arc AD by m. Applying Napier's rules

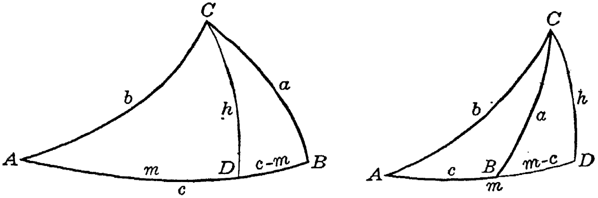


FIG. 106

to the right triangle BDC, we find, from either part of the figure, since $\cos(m - c) = \cos(c - m)$,

$$\begin{aligned} \cos a &= \cos h \cos(c - m) \\ &= \cos h(\cos c \cos m + \sin c \sin m). \end{aligned} \tag{1}$$

From the right triangle ADC, we find

$$\cos b = \cos h \cos m, \qquad \text{or} \qquad \cos m = \frac{\cos b}{\cos h}; \tag{2}$$

and

$$\sin m = \tan h \cot A, \tag{3}$$

$$\sin h = \sin b \sin A. \tag{4}$$

Substituting (2) and (3) in (1), we get

$$\cos a = \cos h\left(\cos c \frac{\cos b}{\cos h} + \sin c \tan h \cot A\right)$$
$$= \cos c \cos b + \sin c \sin h \cot A,$$

or, substituting the value of sin h from (4),

$$\cos a = \cos c \cos b + \sin c \sin b \cos A.$$

Rearranging this formula, and writing the two others obtainable from it by a cyclic change of letters,* we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \qquad (5)$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \qquad (6)$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \qquad (7)$$

These formulas are known as the **law of cosines for sides.**

116. Law of cosines for angles.

Applying formula (5) to $A'B'C'$, the polar triangle of ABC, we get

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'. \qquad (1)$$

If we now make use of the relations between the parts of a triangle and the parts of its polar triangle, $a' = 180° - A$, etc. (see section 106), and of the formulas

$$\cos(180° - \theta) = -\cos\theta, \qquad \sin(180° - \theta) = \sin\theta,$$

(1) reduces to

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a. \qquad (2)$$

Similarly,

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \qquad (3)$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \qquad (4)$$

* See section 54.

The three foregoing formulas constitute the **law of cosines for angles.**

The law of cosines, either for sides or for angles, together with the relations between the parts of a triangle and the parts of its polar triangle, is sufficient for solving any spherical triangle if three parts are given, since it is always possible to find a form of the law which involves the three given parts and a single unknown part. For example, if the given parts are A, B, a, we could use (2) to find C, then (3) and (4) to find b and c respectively. However, the law of cosines is not adapted to the use of logarithms, and as problems of spherical trigonometry ordinarily require accurate results, it is desirable to derive other formulas with which logarithms can be used.

117. Law of tangents.

The law of sines for spherical triangles may be written in the form

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}. \tag{1}$$

By composition and division,*

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\sin a - \sin b}{\sin a + \sin b} \tag{2}$$

Applying formulas (9) and (8) of section 75 (page 132) to the numerator and denominator of the fraction on the left, we reduce it to the form

$$\frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} \tag{3}$$

The right side of (2) may be similarly reduced, and we get the **law of tangents** for spherical triangles,

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}(a + b)} \tag{4}$$

* See the author's *College Algebra*, p. 128.

118. Half-angle formulas.

We shall now develop the half-angle formulas for spherical trigonometry.

From formula (5) of section 74 (page 129), we have *

$$\tan \tfrac{1}{2}A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} \tag{1}$$

Solving equation (5) of the law of cosines (section 115) for $\cos A$, we find

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Subtracting each side from 1, we get

$$\begin{aligned} 1 - \cos A &= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c} \\ &= \frac{\cos(b - c) - \cos a}{\sin b \sin c}. \end{aligned} \tag{2}$$

Similarly, we find

$$1 + \cos A = \frac{\cos a - \cos(b + c)}{\sin b \sin c}. \tag{3}$$

Substituting (2) and (3) in (1), we get

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\cos(b - c) - \cos a}{\cos a - \cos(b + c)}}. \tag{4}$$

By formula (11) of section 75 (page 132),

$$\cos(b - c) - \cos a = -2 \sin \tfrac{1}{2}(b - c + a) \sin \tfrac{1}{2}(b - c - a), \tag{5}$$

$$\cos a - \cos(b + c) = -2 \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(a - b - c). \tag{6}$$

* Only the positive sign is used with the radical, since, by the restriction imposed in section 104, $A < 180°$, and consequently $\tfrac{1}{2}A < 90°$.

If we let *

$$s = \tfrac{1}{2}(a + b + c), \tag{7}$$

then it can easily be shown that

$$\begin{aligned} b + c - a &= 2(s - a),\\ a + c - b &= 2(s - b),\\ a + b - c &= 2(s - c). \end{aligned} \tag{8}$$

By means of (5), (6), (7), we can reduce (4) to the form

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin(s - b)\sin(s - c)}{\sin s \sin(s - a)}}, \tag{9}$$

and, if †

$$\tan r = \sqrt{\frac{\sin(s - a)\sin(s - b)\sin(s - c)}{\sin s}}, \tag{10}$$

(10) reduces to the simpler form

$$\tan \tfrac{1}{2}A = \frac{\tan r}{\sin(s - a)}. \tag{11}$$

Similarly,

$$\tan \tfrac{1}{2}B = \frac{\tan r}{\sin(s - b)}, \tag{12}$$

$$\tan \tfrac{1}{2}C = \frac{\tan r}{\sin(s - c)} \tag{13}$$

These may be termed the **half-angle formulas.**

119. Half-side formulas.

If we solve formula (2) of section 116 for $\cos a$ and proceed somewhat as above, we can derive the **half-side formulas:**

$$\tan \tfrac{1}{2}a = \tan R \cos(S - A), \tag{1}$$

$$\tan \tfrac{1}{2}b = \tan R \cos(S - B), \tag{2}$$

$$\tan \tfrac{1}{2}c = \tan R \cos(S - C), \tag{3}$$

in which ‡

* Cf. section 64.

† It can be shown that r is the radius of the small circle inscribed in the spherical triangle ABC.

‡ It can be shown that R is the radius of the small circle circumscribed about the spherical triangle ABC.

$$\tan R = \sqrt{\frac{-\cos S}{\cos(S - A)\cos(S - B)\cos(S - C)}} \tag{4}$$

and

$$S = \tfrac{1}{2}(A + B + C). \tag{5}$$

This is left as an exercise.

120. Napier's analogies.

Dividing (11) of section 118 by (12) of the same section, we get

$$\frac{\tan \frac{1}{2}A}{\tan \frac{1}{2}B} = \frac{\sin(s - b)}{\sin(s - a)}, \tag{1}$$

and by composition and division,

$$\frac{\tan \frac{1}{2}A - \tan \frac{1}{2}B}{\tan \frac{1}{2}A + \tan \frac{1}{2}B} = \frac{\sin(s - b) - \sin(s - a)}{\sin(s - b) + \sin(s - a)},$$

which reduces as follows:

$$\frac{\dfrac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} - \dfrac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B}}{\dfrac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} + \dfrac{\sin \frac{1}{2}B}{\cos \frac{1}{2}B}} = \frac{2 \cos \frac{1}{2}(2s - a - b) \sin \frac{1}{2}(a - b)}{2 \sin \frac{1}{2}(2s - a - b) \cos \frac{1}{2}(a - b)}$$

$$\frac{\sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c},$$

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c}. \tag{2}$$

Multiplying (9) of section 118 by the corresponding formula for $\tan \frac{1}{2}B$ gives

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = \frac{\sin(s - c)}{\sin s}. \tag{3}$$

Writing the left side in the form $\tan \frac{1}{2}A/\cot \frac{1}{2}B$ and taking steps quite similar to those taken in proving formula (2) of

the present section, we can reduce (3) to the form *

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c}. \tag{4}$$

This is left as an exercise.

It is also left as an exercise to prove, from (2) and (4), by the use of polar triangles, the following formulas:

$$\frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(A - B)}{\cot \frac{1}{2}C}. \tag{5}$$

$$\frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \quad \frac{\tan \frac{1}{2}(A + B)}{\cot \frac{1}{2}C}. \tag{6}$$

By applying cyclic changes to the letters in formulas (2), (4), (5), (6) we obtain eight more formulas, or a total of twelve. These twelve formulas are called **Napier's analogies.**†

121. The six cases.

Problems in the solution of oblique spherical triangles may be classified into the following six cases:

Case I. Three sides given.

Case II. Three angles given.

Case III. Two sides and the included angle given.

Case IV. Two angles and the included side given.

Case V. Two sides and the angle opposite one of them given.

Case VI. Two angles and the side opposite one of them given.

Cases I and II, III and IV, V and VI, are essentially equivalent (in pairs) because of the relations between the parts of a triangle and the parts of its polar triangle. For example, if the three sides of a triangle are given, the three angles of the polar triangle can be found at once, so that

* Formula (4) can also be derived by using the law of tangents and (2).

† The word "analogy" is used in the now obsolete sense of "proportion."

Case I for the given triangle is Case II for the polar triangle.

The six cases can be solved by the application of the half-angle and half-side formulas, Napier's analogies, and the law of sines, as will be illustrated in subsequent sections.

122. Clearing up certain ambiguities.

When Napier's analogies are used, the quadrant in which any part terminates can always be determined by noting the signs of the functions involved. However, when the law of sines is used, two values are found for the required part. Whether one or both of these values are admissible may be determined by the principle established in solid geometry that the three sides and the three angles are in the same order of magnitude (e.g., if $A > B > C$, then $a > b > c$) or by the following theorems:

THEOREM I. *Half the sum of any two sides is in the same quadrant as half the sum of the opposite angles.*

This theorem is easily proved by using Napier's analogy (4), namely,

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \quad \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c}$$

Since each part of a triangle is less than 180°, each of the quantities $\frac{1}{2}(A - B)$ and $\frac{1}{2}c$ is less than 90°. Consequently, $\cos \frac{1}{2}(A - B)$ and $\tan \frac{1}{2}(a - b)$ are both positive. Therefore, $\cos \frac{1}{2}(A + B)$ and $\tan \frac{1}{2}(a + b)$ are of the same sign, and $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ are either both in the first quadrant or both in the second quadrant.

COROLLARY. *If two sides are supplementary the angles opposite are supplementary, and conversely.*

THEOREM II. *A side which differs from 90° more than another side does, terminates in the same quadrant as its opposite angle.*

Suppose, for example, that a differs from 90° more than b does.

From the law of cosines for sides (formula (5) of section 115), we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

From the hypothesis regarding a and b it follows that $\cos a$ is numerically greater than $\cos b$. Moreover, since $\cos c$ is numerically not greater than 1, $\cos a$ is also greater than $\cos b \cos c$. Hence the numerator of the above fraction has the same sign as $\cos a$. The denominator is positive, and consequently $\cos a$ and $\cos A$ have the same sign. Therefore a terminates in the same quadrant as A.

THEOREM III. *An angle which differs from 90° more than another angle does, terminates in the same quadrant as its opposite side.*

This theorem can be proved by using the law of cosines for angles. The proof is left as an exercise.

EXERCISES XVI. A

In the following sets of exercises, A, B, C, are the angles and a, b, c, the sides of spherical triangles.

1. Given $a = 100°$, $b = 95°$, $c = 75°$. State whether the following angles are acute or obtuse: (a) $\frac{1}{2}(A + B)$, (b) $\frac{1}{2}(A + C)$, (c) $\frac{1}{2}(B + C)$.
2. Given $A = 60°$, $B = 100°$, $C = 120°$. State whether the following quantities are acute or obtuse: (a) $\frac{1}{2}(a + b)$, (b) $\frac{1}{2}(a + c)$, (c) $\frac{1}{2}(b + c)$.
3. If $a = 100°$ and $b = 95°$, is A acute or obtuse?
4. Given $a = 100°$, $b = 75°$. Is B acute or obtuse?
5. Given $A = 132°$, $B = 62°$, $C = 42°$. State whether the following sides are acute or obtuse: a, c.
6. Given $A = 76°$, $B = 102°$, $c = 75°$. Which of the following quantities are acute and which obtuse? $\frac{1}{2}(a + b)$, a, $\frac{1}{2}(A + C)$.
7. Given $a = 82°$, $b = 98°$, $c = 99°$. Which of the following angles are acute and which obtuse? $\frac{1}{2}(A + B)$, $\frac{1}{2}(A + C)$, $\frac{1}{2}(B + C)$, A, B, C.

123. Delambre's or Gauss's formulas.

Methods of checking solutions will be given in the model solutions. However, one of the following formulas, known as **Delambre's** or **Gauss's formulas**, always affords a good check, since each formula involves all six parts of the triangle. The formulas are given without proof.

$$\frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}, \tag{1}$$

$$\frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}, \tag{2}$$

$$\frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}c} = \frac{\sin \frac{1}{2}(A + B)}{\cos \frac{1}{2}C}, \tag{3}$$

$$\frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}c} = \frac{\cos \frac{1}{2}(A + B)}{\sin \frac{1}{2}C}. \tag{4}$$

EXERCISE

Deduce Napier's analogies from the foregoing formulas.

124. Solution of Case I.

When we have the *three sides given*, the solution can be effected by the half-angle formulas and checked by the law of sines.

Example.

Solve the triangle $a = 56° 17.2'$, $b = 110° 4.7'$, c $71° 29.3'$.

SOLUTION. $s = \frac{1}{2}(a + b + c)$.

a	$56° 17.2'$
b	$110° 4.7'$
c	$71° 29.3'$
$2s$	$237° 51.2'$
s	$118° 55.6'$
$s - a$	$62° 38.4'$
$s - b$	$8° 50.9'$
$s - c$	$47° 26.3'$
CHECK. s	$118° 55.6'$

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\log \tan r = \tfrac{1}{2}[\log \sin(s-a) + \log \sin(s-b) + \log \sin(s-c) + \operatorname{colog} \sin s].$$

$$\tan\tfrac{1}{2}A = \frac{\tan r}{\sin(s-a)},$$

$$\log \tan\tfrac{1}{2}A = \log \tan r - \log \sin(s-a),$$
etc.

$\log \sin(s-a)$	$9.94848 - 10$
$\log \sin(s-b)$	$9.18701 - 10$
$\log \sin(s-c)$	$9.86720 - 10$
$\operatorname{colog} \sin s$	0.05787
$\log \tan^2 r$	$9.06056 - 10$
$\log \tan r$	$9.53028 - 10$
$\log \tan\frac{1}{2}A$	$9.58180 - 10$
$\log \tan\frac{1}{2}B$	$0.34327 - 10$
$\log \tan\frac{1}{2}C$	$9.66308 - 10$
$\frac{1}{2}A$	$20° 53.7'$
$\frac{1}{2}B$	$65° 35.9'$
$\frac{1}{2}C$	$24° 43.1'$
A	**41° 47.4′**
B	**131° 11.8′**
C	**49° 26.2′**

CHECK. $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = x,$$

$$\log x = \log \sin A - \log \sin a, \text{ etc.}$$

$\log \sin A$	$9.82374 - 10$	$\log \sin B$	$9.87648 - 10$
$\log \sin a$	$9.92004 - 10$	$\log \sin b$	$9.97277 - 10$
$\log$	$9.90370 - 10$	$\log x$	$9.90371 - 10$

$\log \sin C$	$9.88063 - 10$
$\log \sin c$	$9.97692 - 10$
$\log x$	$9.90371 - 10$

125. Solution of Case II.

When we have the *three angles given* the solution can be effected by the half-side formulas and checked by the law of sines.

The computational setup is the same as for Case I.

EXERCISES XVI. B

Solve the following triangles:

1. $a = 125° 40.2'$, $b = 53° 56.2'$, $c = 98° 51.3'$.
2. $a = 63° 24.4'$, $b = 74° 45.2'$, $c = 136° 42.8'$.
3. $a = 53° 42.0'$, $b = 118° 39.5'$, $c = 130° 38.3'$.
4. $a = 158° 33.7'$, $b = 123° 13.5'$, $c = 64° 36.9'$.
5. $a = 84° 35.2'$, $b = 65° 34.4'$, $c = 103° 24.2'$.
6. $A = 105° 14.1'$, $B = 55° 31.4'$, $C = 88° 51.1'$.
7. $A = 43° 40.4'$, $B = 136° 41.5'$, $C = 65° 16.7'$.
8. $A = 63° 24.4'$, $B = 74° 45.2'$, $C = 136° 42.8'$.
9. $A = 128° 17.1'$, $B = 50° 2.5'$, $C = 114° 40.6'$.
10. $A = 81° 52.5'$, $B = 97° 31.1'$, $C = 111° 3.7'$.
11. $a = 51° 43.3'$, $b = 38° 2.4'$, $c = 75° 11.5'$.
12. $a = 146° 48.7'$, $b = 71° 28.1'$, $c = 129° 16.3'$.
13. $A = 83° 54.0'$, $B = 102° 6.4'$, $C = 93° 2.0'$.
14. $A = 143° 35.0'$, $B = 104° 16.2'$, $C = 112° 15.2'$.
15. $a = 170° 30.8'$, $b = 85° 50.4'$, $c = 108° 5.3'$.
16. $a = 69° 8.7'$, $b = 131° 3.9'$, $c = 141° 33.2'$.
17. $A = 128° 15.6'$, $B = 120° 28.2'$, $C = 103° 39.8'$.
18. $A = 59° 4.4'$, $B = 94° 23.2'$, $C = 120° 4.8'$.
19. $A = 45° 24.6'$, $B = 71° 46.4'$, $C = 100° 3.0'$.
20. $a = 105° 27.3'$, $b = 83° 14.7'$, $c = 96° 53.2'$.

126. Solution of Case III.

In this case we have *two sides and the included angle given*. Suppose, for example, that these are a, b, C. We find $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$ from Napier's analogies (6) and (5) respectively (section 120). Angles A and B are then readily found. Side c may then be found by either of Napier's analogies (2) or (4). The solution may be checked

by the law of sines. It is desirable to check angles A and B as soon as they have been found, since they are used in finding c.

Example.

Solve the triangle b 113° 17.3′, $c = 95° 2.5′$, $A = 72° 51.6′$.

SOLUTION.

$$\tan \tfrac{1}{2}(B + C) = \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}(b + c)} \cot \tfrac{1}{2}A,$$

$$\tan \tfrac{1}{2}(B - C) = \frac{\sin \frac{1}{2}(b - c)}{\sin \frac{1}{2}(b + c)} \cot \tfrac{1}{2}A,$$

$$\log \tan \tfrac{1}{2}(B + C) = \log \cos \tfrac{1}{2}(b - c) + \operatorname{colog} \cos \tfrac{1}{2}(b + c) + \log \cot \tfrac{1}{2}A,$$

$$\log \tan \tfrac{1}{2}(B - C) = \log \sin \tfrac{1}{2}(b - c) + \operatorname{colog} \sin \tfrac{1}{2}(b + c) + \log \cot \tfrac{1}{2}A.$$

b	113° 17.3′
c	95° 2.5′
A	72° 51.6′
$b + c$	208° 19.8′
$b - c$	18° 14.8′
$\frac{1}{2}(b + c)$	104° 9.9′
$\frac{1}{2}(b - c)$	9° 7.4′
$\frac{1}{2}A$	36° 25.8′
$\log \cos \frac{1}{2}(b - c)$	9.99447 − 10
$\operatorname{colog} \cos \frac{1}{2}(b + c)$	0.61134 (neg) *
$\log \cot \frac{1}{2}A$	0.13190
$\log \sin \frac{1}{2}(b - c)$	9.20020 − 10
$\operatorname{colog} \sin \frac{1}{2}(b + c)$	0.01341
$\log \tan \frac{1}{2}(B + C)$	0.73771 (neg) *
$\log \tan \frac{1}{2}(B - C)$	9.34551 − 10
$\frac{1}{2}(B + C)$	100° 22.0′
$\frac{1}{2}(B - C)$	12° 29.6′
B	**112° 51.6′**
C	**87° 52.4′**

* The notation (neg) indicates that the corresponding function is negative. Thus, in finding $\frac{1}{2}(B + C)$, we must deduct the value found in the tables

$$\tan \tfrac{1}{2}a = \frac{\sin \frac{1}{2}(B + C)}{\sin \frac{1}{2}(B - C)} \tan \tfrac{1}{2}(b - c),$$

$$\log \tan \tfrac{1}{2}a = \log \sin \tfrac{1}{2}(B + C) + \operatorname{colog} \sin \tfrac{1}{2}(B - C) + \log \tan \tfrac{1}{2}(b - c).$$

$\log \sin \frac{1}{2}(B + C)$	$9.99285 - 10$
$\operatorname{colog} \sin \frac{1}{2}(B - C)$	0.66489
$\log \tan \frac{1}{2}(b - c)$	$9.20572 - 10$
$\log \tan \frac{1}{2}\, a$	$9.86346 - 10$
$\frac{1}{2}a$	$36° \;\; 8.3'$
a	**$72° \; 16.6'$**

CHECK. $\dfrac{\sin A}{\sin a} \quad \dfrac{\sin B}{\sin b} \quad \dfrac{\sin C}{\sin c} = x,$

$$\log x = \log \sin A - \log \sin a, \text{ etc.}$$

$\log \sin A$	$9.98027 - 10$	$\log \sin B$	$9.96447 - 10$
$\log \sin\ a$	$9.97888 - 10$	$\log \sin\ b$	$9.96309 - 10$
$\log x$	0.00139	$\log x$	0.00138

$\log \sin C$	$9.99970 - 10$
$\log \sin\ c$	$9.99832 - 10$
$\log x$	0.00138

127. Solution of Case IV.

The solution of this case, in which we have *two angles and the included side given,* is very similar to the solution of Case III. Using the appropriate analogies of Napier, we find half the sum and half the difference of the required sides. The sides themselves can then be found immediately. The unknown angle is found by using another of Napier's analogies, and the results may be checked by the law of sines, the two sides being checked as soon as they are found.

from 180°, since $\tan \frac{1}{2}(B + C)$ is negative. That is,

$$\tfrac{1}{2}(B + C) = 180° - 79°\, 38.0' = 100°\, 22.0'.$$

This could also be determined by Theorem I of section 122.

Example.

Solve the triangle $A = 93° 14.8'$, $C = 71° 23.2'$, $b = 112° 19.8'$.

SOLUTION.

$$\tan \tfrac{1}{2}(a + c) = \frac{\cos \tfrac{1}{2}(A - C)}{\cos \tfrac{1}{2}(A + C)} \tan \tfrac{1}{2}b,$$

$$\tan \tfrac{1}{2}(a - c) = \frac{\sin \tfrac{1}{2}(A - C)}{\sin \tfrac{1}{2}(A + C)} \tan \tfrac{1}{2}b,$$

$$\log \tan \tfrac{1}{2}(a + c) = \log \cos \tfrac{1}{2}(A - C) + \operatorname{colog} \cos \tfrac{1}{2}(A + C) + \log \tan \tfrac{1}{2}b,$$

$$\log \tan \tfrac{1}{2}(a - c) = \log \sin \tfrac{1}{2}(A - C) + \operatorname{colog} \sin \tfrac{1}{2}(A + C) + \log \tan \tfrac{1}{2}b.$$

A	$93° 14.8'$
C	$71° 23.2'$
b	$112° 19.8'$
$A + C$	$164° 38.0'$
$A - C$	$21° 51.6'$
$\frac{1}{2}(A + C)$	$82° 19.0'$
$\frac{1}{2}(A - C)$	$10° 55.8'$
$\frac{1}{2}b$	$56° 9.9'$
$\log \cos \frac{1}{2}(A - C)$	$9.99205 - 10$
$\operatorname{colog} \cos \frac{1}{2}(A + C)$	0.87388
$\log \tan \frac{1}{2}b$	0.17371
$\log \sin \frac{1}{2}(A - C)$	$9.27786 - 10$
$\operatorname{colog} \sin \frac{1}{2}(A + C)$	0.00392
$\log \tan \frac{1}{2}(a + c)$	1.03964
$\log \tan \frac{1}{2}(a - c)$	$9.45549 - 10$
$\frac{1}{2}(a + c)$	$84° 47.1'$
$\frac{1}{2}(a - c)$	$15° 55.8'$
	$100° 42.9'$
	$68° 51.3'$

$$\cot \tfrac{1}{2}B = \frac{\sin \tfrac{1}{2}(a + c)}{\sin \tfrac{1}{2}(a - c)} \tan \tfrac{1}{2}(A - C),$$

$$\log \cot \tfrac{1}{2}B = \log \sin \tfrac{1}{2}(a + c) + \operatorname{colog} \sin \tfrac{1}{2}(a - c) + \log \tan \tfrac{1}{2}(A - C).$$

$$
\begin{array}{rl}
\log \sin \frac{1}{2}(a+c) & 9.99820-10 \\
\operatorname{colog} \sin \frac{1}{2}(a-c) & 0.56152 \\
\log \tan \frac{1}{2}(A-C) & 9.28581-10 \\
\hline
\log \cot \frac{1}{2}B & 9.84553-10 \\
\frac{1}{2}B & 54^\circ\ 58.9' \\
B & \mathbf{109^\circ\ 57.8'}
\end{array}
$$

CHECK. $\dfrac{\sin A}{\sin a} \quad \dfrac{\sin B}{\sin b} \quad \dfrac{\sin C}{\sin c} = x,$

$$\log x = \log \sin A - \log \sin a, \text{ etc.}$$

$$
\begin{array}{r|l}
\log \sin A & 9.99930-10 \\
\log \sin a & 9.99236-10 \\
\hline
\log x & 0.00694
\end{array}
\qquad
\begin{array}{r|l}
\log \sin B & 9.97309-10 \\
\log \sin b & 9.96615-10 \\
\hline
\log x & 0.00694
\end{array}
$$

$$
\begin{array}{r|l}
\log \sin C & 9.97667-10 \\
\log \sin c & 9.96972-10 \\
\hline
\log x & 0.00695
\end{array}
$$

EXERCISES XVI. C

Solve the following triangles:

1.	$a = 56^\circ\ 19.7'$,	$b = 20^\circ\ 16.7'$,	$C = 114^\circ\ 20.3'$.
2.	$b = 47^\circ\ 29.3'$,	$c = 50^\circ\ 6.3'$,	$A = 129^\circ\ 58.5'$.
3.	$a = 145^\circ\ 58.2'$,	$b = 62^\circ\ 50.6'$,	$C = 134^\circ\ 52.0'$.
4.	$b = 120^\circ\ 30.5'$,	$c = 70^\circ\ 20.3'$,	$A = 50^\circ\ 10.2'$.
5.	$a = 95^\circ\ 12.9'$,	$b = 53^\circ\ 10.1'$,	$C = 49^\circ\ 11.3'$.
6.	$A = 128^\circ\ 36.8'$,	$B = 106^\circ\ 45.2'$,	$c = 87^\circ\ 40.3'$.
7.	$A = 77^\circ\ 59.6'$,	$B = 40^\circ\ 59.8'$,	$c = 108^\circ\ 0.5'$.
8.	$B = 108^\circ\ 28.9'$,	$C = 38^\circ\ 11.5'$,	$a = 52^\circ\ 29.0'$.
9.	$A = 127^\circ\ 19.6'$,	$C = 108^\circ\ 41.5'$,	$b = 125^\circ\ 22.5'$.
10.	$A = 142^\circ\ 30.8'$,	$B = 68^\circ\ 47.7'$,	$c = 135^\circ\ 34.7'$.
11.	$b = 99^\circ\ 40.8'$,	$c = 100^\circ\ 49.5'$,	$A = 65^\circ\ 33.2'$.
12.	$a = 41^\circ\ 5.1'$,	$b = 44^\circ\ 25.4'$,	$C = 37^\circ\ 29.2'$.
13.	$A = 176^\circ\ 16.6'$,	$C = 3^\circ\ 18.2'$,	$b = 27^\circ\ 1.1'$.
14.	$B = 64^\circ\ 48.9'$,	$C = 40^\circ\ 23.3'$,	$a = 108^\circ\ 39.2'$.
15.	$a = 88^\circ\ 37.7'$,	$b = 125^\circ\ 18.3'$,	$C = 102^\circ\ 16.6'$.
16.	$a = 67^\circ\ 12.6'$,	$c = 135^\circ\ 0.9'$,	$B = 74^\circ\ 45.2'$.
17.	$A = 34^\circ\ 29.5'$,	$B = 36^\circ\ 6.8'$,	$c = 85^\circ\ 59.0'$.

18. $A = 78° 30.8'$, $B = 91° 28.2'$, $c = 51° 22.4'$.
19. $a = 132° 46.7'$, $b = 59° 50.1'$, $C = 56° 28.4'$.
20. $b = 28° 20.3'$, $c = 112° 1.9'$, $A = 79° 28.6'$.

128. Solution of Case V.

Case V, in which we have *two sides and the angle opposite one of them given,* presents the same peculiarities as the corresponding case in plane trigonometry. Suppose that the given parts are a, b, A. Angle B can be determined by the law of sines,

$$\sin B = \frac{\sin b \sin A}{\sin a} \tag{1}$$

If the ratio on the right of this equation is greater than 1 (in other words, if $\log \sin B > 0$), no solution exists.

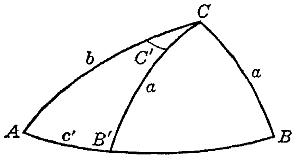


FIG. 107

If this ratio is equal to 1, B is 90° and the resulting right triangle is a unique solution.

If the ratio is less than 1, we find two values for B, the tabular value and its supplement. In this event there may be two solutions (see Fig. 107). The number of solutions may be determined by the principles of section 122.

The remaining angle, and likewise the required side, can be found by using appropriate forms of Napier's analogies.

Checking is perhaps best done by means of one of Delambre's formulas. Suppose, for example, that we rewrite (1) of section 123 in the form

$$\frac{\sin \frac{1}{2}(a - b) \cos \frac{1}{2}C}{\sin \frac{1}{2}(A - B) \sin \frac{1}{2}c} = 1. \tag{2}$$

Then, the logarithm of the left side should be equal to zero (since $\log 1 = 0$) if the work is correct.

Example.

Solve the triangle $a = 100° \, 48.2'$, $b = 70° \, 11.4'$, $B = 71° \, 9.6'$.

SOLUTION.

$$\sin A = \frac{\sin a \sin B}{\sin b},$$

$\log \sin A = \log \sin a + \log \sin B + \operatorname{colog} \sin b.$

a	$100° \, 48.2'$
b	$70° \, 11.4'$
B	$71° \quad 9.6'$
$\log \sin a$	$9.99223 - 10$
$\log \sin B$	$9.97608 - 10$
$\operatorname{colog} \sin b$	0.02649
$\log \sin A$	$9.99480 - 10$
A	$\mathbf{81° \, 9.0'}$, $A' = \mathbf{98° \, 51.0'}$

$$\cot \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B),$$

$$\log \cot \tfrac{1}{2}C = \log \sin \tfrac{1}{2}(a + b) + \operatorname{colog} \sin \tfrac{1}{2}(a - b) + \log \tan \tfrac{1}{2}(A - B).$$

$a + b$	$170° \, 59.6'$
$a - b$	$30° \, 36.8'$
$A + B$	$152° \, 18.6'$, $A' + B = 170° \quad 0.6'$
$A - B$	$9° \, 59.4'$, $A' - B = 27° \, 41.4'$
$\frac{1}{2}(a + b)$	$85° \, 29.8'$
$\frac{1}{2}(a - b)$	$15° \, 18.4'$
$\frac{1}{2}(A + B)$	$76° \quad 9.3'$, $\frac{1}{2}(A' + B) = 85° \quad 0.3'$
$\frac{1}{2}(A - B)$	$4° \, 59.7'$, $\frac{1}{2}(A' - B) = 13° \, 50.7'$
$\log \tan \frac{1}{2}(A - B)$	$8.94151 - 10$
$\log \sin \frac{1}{2}(a + b)$	$9.99866 - 10$
$\operatorname{colog} \sin \frac{1}{2}(a - b)$	0.57842
$\log \tan \frac{1}{2}(A' - B)$	$9.39174 - 10$
$\log \cot \frac{1}{2}C$	$9.51859 - 10$
$\log \cot \frac{1}{2}C'$	$9.96882 - 10$
$\frac{1}{2}C$	$71° \, 44.0'$
$\frac{1}{2}C'$	$47° \quad 3.3'$
C	$\mathbf{143° \, 28.0'}$
C'	$\mathbf{94° \quad 6.6'}$

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b).$$

$$\log\tan\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A+B) + \operatorname{colog}\sin\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}(A-B).$$

$\log\sin\frac{1}{2}(A+B)$	$9.98720-10$
$\operatorname{colog}\sin\frac{1}{2}(A-B)$	1.06014
$\log\tan\frac{1}{2}(a-b)$	$9.43727-10$
$\log\sin\frac{1}{2}(A'+B)$	$9.99835-10$
$\operatorname{colog}\sin\frac{1}{2}(A'-B)$	0.62106
$\log\tan\frac{1}{2}c$	0.48461
$\log\tan\frac{1}{2}c'$	0.05668
$\frac{1}{2}c$	$71°\ 51.6'$
$\frac{1}{2}c'$	$48°\ 43.7'$
c	**$143°\ 43.2'$**
c'	**$97°\ 27.4'$**

CHECK. 1st solution.

$$\frac{\sin\frac{1}{2}(a-b)\cos\frac{1}{2}C}{\sin\frac{1}{2}(A-B)\sin\frac{1}{2}c} = 1,$$

$$\log\sin\tfrac{1}{2}(a-b) + \log\cos\tfrac{1}{2}C + \operatorname{colog}\sin\tfrac{1}{2}(A-B) + \operatorname{colog}\sin\tfrac{1}{2}c = 0.$$

$\log\sin\frac{1}{2}(a-b)$	$9.42158-10$
$\log\cos\frac{1}{2}C$	$9.49615-10$
$\operatorname{colog}\sin\frac{1}{2}(A-B)$	1.06014
$\operatorname{colog}\sin\frac{1}{2}c$	0.02214
	0.00001

129. Solution of Case VI.

Case VI, *two angles and the side opposite one of them given*, is so similar to Case V that we shall not give a detailed discussion. A model solution, however, will be given.

Example.

Solve the triangle $A = 121°\ 17.7'$, $B = 29°\ 7.7'$, $a = 136°\ 12.0'$.

SOLUTION. $$\sin b = \frac{\sin a \sin B}{\sin A}$$

$$\log\sin b = \log\sin a + \log\sin B + \operatorname{colog}\sin A.$$

A	$121° \; 17.7'$
B	$29° \;\; 7.7'$
	$136° \; 12.0'$
$\log \sin \; a$	$9.84020 - 10$
$\log \sin B$	$9.68732 - 10$
$\operatorname{colog} \sin A$	0.06829
$\log \sin \; b$	$9.59581 - 10$
b	$\mathbf{23° \; 13.3'}, \; b' = 156° \; 46.7'$

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b),$$

$$\log \tan \tfrac{1}{2}c = \log \sin \tfrac{1}{2}(A + B) + \operatorname{colog} \sin \tfrac{1}{2}(A - B) + \log \tan \tfrac{1}{2}(a - b).$$

$A + B$	$150° \; 25.4'$
$A - B$	$92° \; 10.0'$
$a + b$	$159° \; 25.3'$
$a - b$	$112° \; 58.7'$
$\frac{1}{2}(A + B)$	$75° \; 12.7'$
$\frac{1}{2}(A - B)$	$46° \;\; 5.0'$
$\frac{1}{2}(a + b)$	$79° \; 42.6'$
$\frac{1}{2}(a - b)$	$56° \; 29.4'$
$\log \; \sin \frac{1}{2}(A + B)$	$9.98537 -$
$\operatorname{colog} \; \sin \frac{1}{2}(A - B)$	0.14246
$\log \tan \frac{1}{2}(a - b)$	0.17905
$\log \tan \frac{1}{2}c$	0.30688
$\frac{1}{2}c$	$63° \; 44.5'$
c	$\mathbf{127° \; 29.0'}$

$$\cot \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B),$$

$$\log \cot \tfrac{1}{2}C = \log \sin \tfrac{1}{2}(a + b) + \operatorname{colog} \sin \tfrac{1}{2}(a - b) + \log \tan \tfrac{1}{2}(A - B).$$

$\log \; \sin \frac{1}{2}(a + b)$	$9.99296 - 10$
$\operatorname{colog} \; \sin \frac{1}{2}(a - b)$	0.07894
$\log \tan \frac{1}{2}(A - B)$	0.01643
$\log \cot \frac{1}{2}C$	0.08833
$\frac{1}{2}C$	$39° \; 12.8'$
C	$\mathbf{78° \; 25.6'}$

* Not admissible; for $A > B$, and therefore a must be greater than b.

CHECK. $$\frac{\sin \frac{1}{2}(a - b) \cos \frac{1}{2}C}{\sin \frac{1}{2}(A - B) \sin \frac{1}{2}c} = 1,$$

$$\log \sin \tfrac{1}{2}(a - b) + \log \cos \tfrac{1}{2}C + \text{colog} \sin \tfrac{1}{2}(A - B) + \text{colog} \sin \tfrac{1}{2}c = 0.$$

$\log \sin \frac{1}{2}(a - b)$	$9.92106 - 10$
$\log \cos \frac{1}{2}C$	$9.88919 - 10$
$\text{colog} \sin \frac{1}{2}(A - B)$	0.14246
$\text{colog} \sin \frac{1}{2}c$	0.04730
	0.00001

EXERCISES XVI. D

Solve the following triangles:

1. $a = 44° 48.3'$, $b = 17° 36.7'$, $A = 63° 24.8'$.
2. $a = 56° 30.0'$, $b = 31° 20.0'$, $A = 105° 11.2'$.
3. $a = 52° 45.3'$, $b = 71° 12.7'$, $A = 46° 22.2'$.
4. $b = 68° 52.8'$, $c = 56° 49.8'$, $C = 45° 15.2'$.
5. $a = 30° 38.1'$, $c = 31° 29.8'$, $A = 87° 53.3'$.
6. $A = 109° 20.2'$, $B = 134° 16.4'$, $a = 148° 48.7'$.
7. $A = 143° 17.4'$, $B = 70° 18.4'$, $a = 160° 40.6'$.
8. $A = 61° 37.9'$, $B = 139° 54.6'$, $b = 150° 17.4'$.
9. $A = 70° 15.2'$, $B = 119° 43.8'$, $b = 80° 24.4'$.
10. $B = 24° 30.5'$, $C = 61° 29.5'$, $c = 34° 0.5'$.
11. $a = 80° 5.3'$, $b = 82° 4.0'$, $A = 83° 34.2'$.
12. $a = 134° 15.9'$, $b = 150° 57.1'$, $B = 144° 22.7'$.
13. $A = 79° 37.3'$, $C = 145° 52.2'$, $c = 150° 42.7'$.
14. $A = 60° 20.2'$, $B = 17° 12.9'$, $b = 43° 50.5'$.
15. $a = 148° 34.4'$, $b = 142° 11.6'$, $A = 153° 17.6'$.
16. $a = 40° 20.4'$, $b = 20° 18.2'$, $A = 60° 44.4'$.
17. $A = 117° 54.4'$, $B = 45° 8.6'$, $a = 76° 37.5'$.
18. $b = 119° 19.9'$, $c = 160° 2.3'$, $C = 139° 9.1'$.
19. $A = 104° 40.0'$, $B = 80° 13.6'$, $a = 126° 50.4'$.
20. $a = 40° 5.4'$, $b = 118° 22.1'$, $A = 29° 42.6'$.

130. Summary of methods.

The methods of solving oblique spherical triangles are epitomized below.

Case I. Three sides given.	Use **half-angle formulas.** Check by law of sines.
Case II. Three angles given.	Use **half-side formulas.** Check by law of sines.
Case III. Two sides and the included angle given.	Find half the sum and half the difference of the required angles by using appropriate forms of **Napier's analogies.** The required angles are then readily found. Find required side by another of **Napier's analogies.** Check by law of sines.
Case IV. Two angles and the included side given.	Find half the sum and half the difference of the required sides by using appropriate forms of **Napier's analogies.** The required sides are then readily found. Find required angle by another of **Napier's analogies.** Check by law of sines.
Case V. Two sides and the angle opposite one of them given.	Use **law of sines** to find an angle. Find remaining angle and required side by appropriate forms of **Napier's analogies.** Note number of solutions. Check by one of Delambre's formulas.
Case VI. Two angles and the side opposite one of them given.	Use **law of sines** to find a side. Find remaining side and required angle by appropriate forms of **Napier's analogies.** Note number of solutions. Check by one of Delambre's formulas.

MISCELLANEOUS EXERCISES XVI. E

Solve the following triangles:

1. $a = 18° 29.3'$, $b = 30° 37.1'$, $C = 52° 51.8'$.

2. $a = 114° 43.3'$, $b = 136° 19.6'$, $c = 43° 18.5'$.

3. $A = 33° 15.1'$,	$B = 31° 34.6'$,	$C = 161° 25.3'$.
4. $A = 80° 2.3'$,	$a = 118° 20.3'$,	$b = 69° 56.3'$.
5. $B = 140° 43.2'$,	$C = 100° 4.6'$,	$a = 60° 43.6'$.
6. $a = 76° 40.4'$,	$b = 54° 21.3'$,	$c = 36° 8.7'$.
7. $a = 148° 34.4'$,	$b = 142° 11.6'$,	$A = 153° 17.6'$.
8. $A = 40° 20.4'$,	$a = 60° 44.4'$,	$b = 20° 18.2'$.
9. $a = 103° 44.7'$,	$b = 64° 12.3'$,	$C = 98° 33.8'$.
10. $A = 30° 51.2'$,	$B = 71° 36.0'$,	$C = 90°$.
11. $A = 100° 51.3'$,	$B = 80° 47.6'$,	$C = 74° 3.3'$.
12. $A = 150° 47.0'$,	$C = 98° 22.7'$,	$c = 90°$.
13. $A = 64° 34.3'$,	$B = 119° 54.6'$,	$C = 63° 20.2'$.
14. $A = 104° 30.7'$,	$B = 62° 52.1'$,	$c = 56° 6.4'$.
15. $A = 117° 54.4'$,	$B = 45° 8.6'$,	$a = 76° 37.5'$.
16. $C = 50° 10.2'$,	$b = 69° 34.9'$,	$c = 120° 30.5'$.
17. $C = 50° 10.2'$,	$b = 120° 30.5'$,	$c = 69° 34.9'$.
18. $A = 92° 47.4'$,	$B = 73° 1.3'$,	$c = 26° 6.9'$.
19. $a = 80° 39.1'$,	$b = 75° 12.3'$,	$c = 141° 5.6'$.
20. $A = 61° 37.9'$,	$C = 139° 54.6'$,	$c = 150° 17.4'$.
21. $A = 53° 15.5'$,	$C = 68° 58.5'$,	$b = 67° 12.6'$.
22. $A = 99° 34.1'$,	$B = 67° 46.7'$,	$C = 91° 56.8'$.
23. $a = 41° 19.3'$,	$b = 112° 36.2'$,	$c = 78° 9.6'$.
24. $a = 58° 49.6'$,	$b = 75° 12.1'$,	$C = 102° 58.0'$.
25. $A = 104° 30.7'$,	$B = 62° 52.1'$,	$c = 56° 6.4'$.
26. $A = 32° 40.2'$,	$B = 122° 11.1'$,	$C = 42° 36.2'$.
27. $A = 104° 40.0'$,	$B = 80° 13.6'$,	$a = 126° 50.4'$.
28. $A = 65° 33.2'$,	$b = 99° 40.8'$,	$c = 100° 49.5'$.
29. $A = 113° 30.0'$,	$B = 125° 31.6'$,	$a = 66° 44.7'$.
30. $B = 10° 10.2'$,	$C = 90°$,	$b = 10° 10.2'$.

31. Find the perimeter and the area of the spherical triangle in which $A = 65° 50'$, $b = 63° 17'$, $c = 107° 23'$, the radius of the sphere being 5 inches.

32. A triangle whose sides are 100°, 50°, and 60° lies on a sphere of radius 10 inches. Find the difference between the area of this triangle and that of an equilateral triangle having the same perimeter.

33. A triangle whose angles are 100°, 50°, and 60° lies on a sphere of radius 10 inches. Find the difference between the perimeter of this triangle and that of an equiangular triangle having the same area.

CHAPTER XVII

Applications of Spherical Trigonometry

131. Terrestrial sphere.

In long distance measurements on the surface of the earth, and in navigation, the earth is treated as a sphere having a radius of 3959 miles. This is called the **terrestrial sphere.**

It rotates about a diameter, called its **axis,** which pierces the sphere in the **north pole** P and the **south pole** P'. (See Fig. 108.)

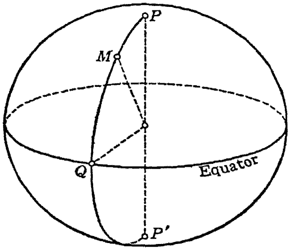


FIG. 108

The **equator** is the great circle whose plane is perpendicular to the axis.

A **meridian** is a great circle passing through the poles, for example, PMQ.

The **latitude** of a point M is the angular distance of the point from the equator, and will be considered positive if the point is north of the equator, negative if the point is south of the equator. It is measured by the arc QM of the meridian through the point. The **colatitude** is 90° minus the latitude.* It is the angular distance from the north pole and is measured by the arc MP.

The meridian through Greenwich is called the **prime meridian.** The **longitude** of a point is the angle between the prime meridian and the meridian through the point. It is measured by the number of degrees in the arc intercepted

* If the point is south of the equator, say 30° south, its latitude is $-30°$ and its colatitude is $90° - (-30°) = 120°$.

at the equator by these two meridians.* If for example, in Fig. 109, PGG' is the prime meridian and PAA' is the meridian through the point A, these meridians cutting the equator in G' and A' respectively, then the longitude of A is measured by the number of degrees in the arc $G'A'$. Longitude will be considered positive if the point is west of the prime meridian and negative if the point is east.

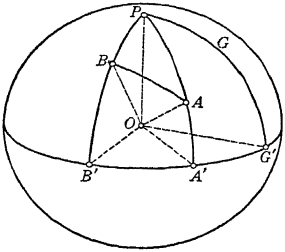


FIG. 109

The **distance** between two points A and B is the length of the arc AB (not greater than a semicircumference) of a great circle passing through A and B. This distance may be expressed in angular measure or in linear measure. To convert from angular units to linear units, we note that a **nautical mile** is the length of one minute of arc of a great circle on the terrestrial sphere. This is about 1.1516 **statute miles** of 5280 feet each, or 6080 feet.†

The **bearing** of point B from point A is the angle which the arc AB makes with the meridian through A (angle PAB in Fig. 109).‡

132. Terrestrial triangle.

To find the distance between A and B, and their bearings from each other, we consider the **terrestrial triangle** ABP, whose vertices are the two points and the north pole. If the latitude and longitude of the points are given, we can find arcs AP and BP, also angle APB, immedi-

* It is also frequently expressed in hours, minutes, and seconds of time (cf. section 133), 1 hour being equivalent to 1/24 of 360°, or 15° of arc, 1 minute of time consequently being equivalent to 15 minutes of arc, and 1 second of time to 15 seconds of arc.

† The United States nautical mile is 6080.27 feet, the British nautical mile is 6080 feet.

‡ In the United States Navy bearings are measured from 0° to 360°, from north through east. According to this convention, the bearing of B from A in Fig. 109 would be found by subtracting angle PAB from 360°.

ately, so that we have a problem under Case III, namely, two sides and the included angle given.

Example.

Find the distance between New York (40° 43′ N, 74° 0′ W) and Liverpool (53° 24′ N, 3° 4′ W) and the bearing of each of these places from the other.

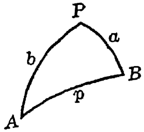


FIG. 110

SOLUTION. Represent New York by A and Liverpool by B (Fig. 110). Then,

$$b = AP = \text{colatitude } A = 90° - 40° 43' = 49° 17',$$

$$a = BP = \text{colatitude } B = 90° - 53° 24' = 36° 36',$$

$$P = \text{difference in longitude} = 74° 0' - 3° 4' = 70° 56'.$$

$$\tan \tfrac{1}{2}(B + A) \quad \frac{\cos \frac{1}{2}(b - a)}{\cos \frac{1}{2}(b + a)} \cot \tfrac{1}{2}P,$$

$$\tan \tfrac{1}{2}(B - A) = \frac{\sin \frac{1}{2}(b - a)}{\sin \frac{1}{2}(b + a)} \cot \tfrac{1}{2}P,$$

$$\log \tan \tfrac{1}{2}(B + A) = \log \cos \tfrac{1}{2}(b - a) + \operatorname{colog} \cos \tfrac{1}{2}(b + a) + \log \cot \tfrac{1}{2}P,$$

$$\log \tan \tfrac{1}{2}(B - A) = \log \sin \tfrac{1}{2}(b - a) + \operatorname{colog} \sin \tfrac{1}{2}(b + a) = \log \cot \tfrac{1}{2}P.$$

$b + a$	85° 53′
$b - a$	12° 41′
$\frac{1}{2}(b + a)$	42° 56.5′
$\frac{1}{2}(b - a)$	6° 20.5′
$\frac{1}{2}P$	35° 28′
$\log \cos \frac{1}{2}(b - a)$	9.99734 − 10
$\operatorname{colog} \cos \frac{1}{2}(b + a)$	0.13546
$\log \cot \frac{1}{2}P$	0.14727
$\log \sin \frac{1}{2}(b - a)$	9.04319 − 10
$\operatorname{colog} \sin \frac{1}{2}(b + a)$	0.16669
$\log \tan \frac{1}{2}(B + A)$	0.28007
$\log \tan \frac{1}{2}(B - A)$	9.35715 − 10
$\frac{1}{2}(B + A)$	62° 19′
$\frac{1}{2}(B - A)$	12° 49′
B	75° 8′
A	49° 30′

$$\tan \tfrac{1}{2}p = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a).$$

$$\begin{aligned} \log \tan \tfrac{1}{2}p &= \log \sin \tfrac{1}{2}(B + A) \\ &\quad + \operatorname{colog} \sin \tfrac{1}{2}(B - A) + \log \tan \tfrac{1}{2}(b - a). \end{aligned}$$

$$\begin{array}{r|l} \log \sin \frac{1}{2}(B + A) & 9.94720 - 10 \\ \operatorname{colog} \sin \frac{1}{2}(B - A) & 0.65398 \\ \log \tan \frac{1}{2}(b - a) & 9.04586 - 10 \\ \hline \log \tan \frac{1}{2}p & 9.64704 - 10 \\ \frac{1}{2}p & 23^\circ\ 55' \\ p & 47^\circ\ 50' = 2870' \end{array}$$

Distance = 2870 nautical miles.
Bearing of Liverpool from New York = $A = N\ 49^\circ\ 30'$ E.
Bearing of New York from Liverpool = $B = N\ 75^\circ\ 8'\ W$.

The solution should be checked by the law of sines.

EXERCISES XVII. A

Find the distances between the following places, also the bearing of each from the other. Latitudes and longitudes are given at the end of the set of exercises.

1. New York and San Francisco.
2. New York and Paris.
3. New York and Cape of Good Hope.
4. San Francisco and Sydney.
5. San Francisco and Rio de Janeiro.
6. New York and Rio de Janeiro.
7. Rio de Janeiro and Sydney.
8. Moscow and San Francisco.
9. How close to the north pole does the great circle path of the preceding exercise pass?
10. A ship sailed due east from New York to a point on the meridian of 10° W near Portugal. Find the distance it would have saved if it had sailed along the arc of a great circle.
11. A ship sails from New York to Cape of Good Hope along the arc of a great circle. Find its course (i.e., direction) (a) when it crosses the equator, (b) when it crosses the meridian of 10° W. (Use results of exercise 3.)
12. Find the area of the triangle whose vertices are New York,

San Francisco, and Rio de Janeiro. (Use results of exercises 1, 5, 6.)

13. An airplane flies from New York to Chicago in 3 hours and 45 minutes. What is its average rate of speed in statute miles per hour?

14. An airplane flew from Chicago to San Francisco at an average speed of 180 statute miles per hour. How long did the flight take?

	Latitude	Longitude
Cape of Good Hope	34° 21′ S	18° 30′ E
Chicago	41° 50′ N	87° 37′ W
Moscow	55° 45′ N	37° 34′ E
New York	40° 43′ N	74° 0′ W
Paris	48° 50′ N	2° 20′ E
Rio de Janeiro	22° 54′ S	43° 10′ W
San Francisco	37° 47′ N	122° 26′ W
Sydney	33° 52′ S	151° 12′ E

133. Celestial sphere.

A sphere, concentric with the earth, and having a radius of indefinite length, is called the **celestial sphere.** (See

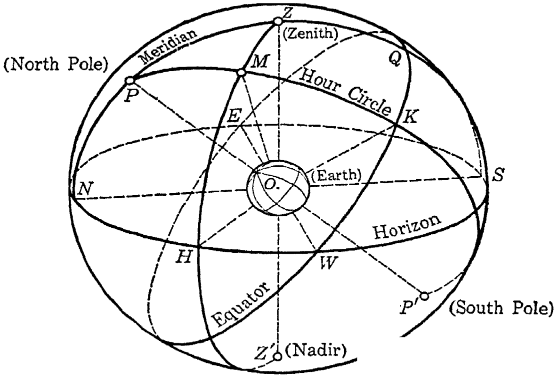


Fig. 111

Fig. 111, in which the earth is located at the point O.) With any point on this sphere is associated a direction, and thus

the angular distance (although not a linear distance) between any two points on it may be considered.

The points where the axis of the earth intersects the celestial sphere are the **north** and **south celestial poles,** P and P', respectively.

The plane of the equator of the earth cuts the celestial sphere in the **celestial equator,** EQW.

Great circles, such as PMP', passing through th celestial poles are called **hour circles.** The hour circle of the observer, the great circle $NPZQS$ in the figure, is called the observer's **celestial meridian.**

The point Z on the celestial sphere vertically above the observer is called the **zenith** of the observer. The diametrically opposite point, Z', is called the **nadir.**

The **horizon** of the observer is the great circle $NESW$ having the zenith and nadir as poles. On the horizon the cardinal points (north, south, east, west) are marked by the respective initial letters.

The **declination** of a star or other heavenly body, whose projection on the celestial sphere is represented by M in the figure, is its angular distance north or south of the celestial equator. It is regarded as positive if the body is north of the equator, negative if the body is south. The declination of the body M in Fig. 111 is measured by the arc KM of the hour circle of the body. Declination corresponds to latitude on the earth.

The **hour angle** of the body M is the angle at the pole between the celestial meridian (i.e., the hour circle of the observer) and the hour circle through the body. It is the angle ZPM in the figure, and may be measured by the arc QK of the celestial equator. It is usually measured from the celestial meridian, toward the west, from 0° to 360° or from 0 to 24 hours. Since the celestial sphere apparently rotates through 360° in 24 hours, 1 hour corresponds to $\frac{1}{24} \times 360° = 15°$, and we have the following relations between measures of time and angular measure:

$$\begin{aligned} 1 \text{ hour} &= 15 \text{ degrees} && (1^h = 15°), \\ 1 \text{ minute of time} &= 15 \text{ minutes of arc} && (1^m = 15'), \\ 1 \text{ second of time} &= 15 \text{ seconds of arc} && (1^s = 15''). \end{aligned}$$

The **altitude** of the body M is its distance above the horizon, and is measured by the arc HM.* The altitude is taken as positive if the body is above the horizon, negative if it is below.

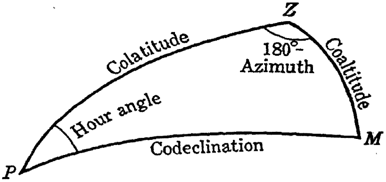


FIG. 112

The **azimuth** of the body is the angle at the zenith between the celestial meridian $PZQS$ and the great circle $ZMHZ'$ through the zenith and the body. It may be measured from north or from south. If, for example, it is measured from the south, the azimuth of M in Fig. 111 is the angle SZM.

A heavenly body may be located by its declination and its hour angle, or by its altitude and azimuth.

134. Astronomical triangle.

The spherical triangle PZM whose vertices are the celestial pole,† the zenith, and the projection of a heavenly body on the celestial sphere, is called the **astronomical triangle.**

A study of Fig. 111 shows that

$$ZM = \textbf{coaltitude}, \tag{1}$$

$$MP = \textbf{codeclination}, \tag{2}$$

$$PZ = \textbf{colatitude}, \tag{3}$$

where the prefix "co" obviously denotes "complement of." Moreover,

$$P = \textbf{hour angle}, \tag{4}$$

$$Z = 180° - \textbf{azimuth}. \tag{5}$$

The angle M is of no special interest.

* It can easily be shown that the altitude of the north celestial pole, at any place of observation, is the latitude of the place.

† The north pole if the observer is in the northern hemisphere, the south pole if he is in the southern hemisphere.

If any three of the other five parts are known, the remaining two can be found. Thus, if an observer knows his latitude, and measures the altitude and azimuth of the sun, he can find PZ, ZM, and Z. From these he can compute the hour angle P. This would give the **local apparent time** (shown on a sundial).

From the American Nautical Almanac or the American Air Almanac (these are published by the United States Naval Observatory) can be obtained the declination of each of many heavenly bodies (sun, moon, planets, and several hundred stars) for any hour of the day. If an observer knows the time and measures the altitude of the sun, he has, after finding the declination of the heavenly body M from the Almanac, the values of ZM, MP, and P, from which he can compute PZ and hence his latitude.

Example 1.

An observation taken in St. Louis (latitude 38° 38′ N) showed the altitude of the sun to be 30° 30′. Its declination was found to be 10° 20′ N. What was the time of day?

Solution. In the astronomical triangle we have

$$\begin{aligned} m &= \text{colat.} = 90° - 38°\,38' = 51°\,22', \\ p &= \text{coalt.} = 90° - 30°\,30' = 59°\,30', \\ z &= \text{codec.} = 90° - 10°\,20' = 79°\,40'. \end{aligned}$$

This is Case I. Since only one angle is required, we use formula (9) of section 118 (page 216).

$$s = \tfrac{1}{2}(m + p + z).$$

$$\tan \tfrac{1}{2}P = \sqrt{\frac{\sin(s - m)\sin(s - z)}{\sin s \sin(s - p)}},$$

$$\log \tan \tfrac{1}{2}P$$
$$= \tfrac{1}{2}[\log\sin(s-m)+\log\sin(s-z)+\operatorname{colog}\sin s+\operatorname{colog}\sin(s-p)].$$

	$2s$	$190°\ 32'$
	s	$95°\ 16'$
	$s - m$	$43°\ 54'$
	$s - p$	$35°\ 46'$
	$s - z$	$15°\ 36'$
CHECK.	s	$95°\ 16'$
	$\log \sin(s - m)$	$9.84098 - 10$
	$\log \sin(s - z)$	$9.42962 - 10$
	$\operatorname{colog} \sin s$	0.00184
	$\operatorname{colog} \sin(s - p)$	0.23323
	$\log \tan^2 \frac{1}{2}P$	$9.50567 - 10$
	$\log \tan \frac{1}{2}P$	$9.75284 - 10$
	$\frac{1}{2}P$	$29°\ 30.7'$
	P	$59°\ \ 1'$

Reducing the hour angle P to units of time (see section 133), we get $P = 59°\ 1' \div 15 = 3^h\ 56^m$. If the observation was taken in the afternoon, the time was 3:56 p.m. If the observation was taken in the morning, the time was $12^h - 3^h\ 56^m = 8^h\ 4^m$, or 8:04 a.m. In either case the time is local apparent time.

Example 2.

The declination of a star is $7°\ 54'$ N, its hour angle is $48°\ 51'$. Find its azimuth, it being given that the observer is in latitude $67°\ 49'$ N.

SOLUTION. In the astronomical triangle we have

$$\begin{aligned} z &= \text{codec.} = 90° - 7°\ 54' = 82°\ 6', \\ P &= \text{hr.} \angle = 48°\ 51', \\ m &= \text{colat.} = 90° - 67°\ 49' = 22°\ 11'. \end{aligned}$$

This is Case III.

$$\tan \tfrac{1}{2}(Z + M) = \frac{\cos \frac{1}{2}(z - m)}{\cos \frac{1}{2}(z + m)} \cot \tfrac{1}{2}P,$$

$$\tan \tfrac{1}{2}(Z - M) = \frac{\sin \frac{1}{2}(z - m)}{\sin \frac{1}{2}(z + m)} \cot \tfrac{1}{2}P,$$

$$\begin{aligned} \log \tan \tfrac{1}{2}(Z + M) = {} & \log \cos \tfrac{1}{2}(z - m) \\ & + \operatorname{colog} \cos \tfrac{1}{2}(z + m) + \log \cot \tfrac{1}{2}P, \end{aligned}$$

$$\log \tan \tfrac{1}{2}(Z - M) = \log \sin \tfrac{1}{2}(z - m) + \operatorname{colog} \sin \tfrac{1}{2}(z + m) + \log \cot \tfrac{1}{2}P.$$

$z + m$	$104° 17'$
$z - m$	$59° 55'$
$\frac{1}{2}(z + m)$	$52° 8.5'$
$\frac{1}{2}(z - m)$	$29° 57.5'$
$\frac{1}{2}P$	$24° 25.5'$
$\log \cos \frac{1}{2}(z - m)$	$9.93772 - 10$
$\operatorname{colog} \cos \frac{1}{2}(z + m)$	0.21204
$\log \cot \frac{1}{2}P$	0.34280
$\log \sin \frac{1}{2}(z - m)$	$9.69842 - 10$
$\operatorname{colog} \sin \frac{1}{2}(z + m)$	0.10263
$\log \tan \frac{1}{2}(Z + M)$	0.49256
$\log \tan \frac{1}{2}(Z - M)$	0.14385
$\frac{1}{2}(Z + M)$	$72° 10.0'$
$\frac{1}{2}(Z - M)$	$54° 19.2'$
Z	$126° 29.2'$
M	$17° 50.8'$

$$\text{Azimuth} = 180° - Z = 53° 31'.$$

CHECK.
$$\frac{\sin Z}{\sin z} = \frac{\sin M}{\sin m} = x,$$

$$\log x = \log \sin Z - \log \sin z = \log \sin M - \log \sin m.$$

$\log \sin Z$	$9.90525 - 10$
$\log \sin z$	$9.99586 - 10$
$\log x$	$9.90939 - 10$
$\log \sin M$	$9.48639 - 10$
$\log \sin m$	$9.57700 - 10$
$\log x$	$9.90939 - 10$

Example 3.

An observer in the northern hemisphere finds the altitude of the sun to be 35° 23′ at 9:15 a.m., local apparent time. If the declination of the sun is 10° 48′ S, what is the latitude of the place of observation?

SOLUTION. In the astronomical triangle we have

$$z = MP = \text{codec.} = 90^\circ + 10^\circ\,48' = 100^\circ\,48',$$
$$p = ZM = \text{coalt.} = 90^\circ - 35^\circ\,23' = 54^\circ\,37',$$
$$P = \text{hr. } \angle = 12^\text{h} - 9^\text{h}\,15^\text{m} = 2^\text{h}\,45^\text{m} = 41^\circ\,15'.$$

This is Case V.

$$\sin Z = \frac{\sin z \sin P}{\sin p}.$$

$$\log \sin Z = \log \sin z + \log \sin P + \operatorname{colog} \sin p.$$

$\log \sin z$	$9.99224 - 10$
$\log \sin P$	$9.81911 - 10$
$\operatorname{colog} \sin p$	0.08868
$\log \sin Z$	$9.90003 - 10$
	$52^\circ\,36'$ * or $127^\circ\,24'$

$$\tan \tfrac{1}{2}m = \frac{\sin \frac{1}{2}(Z + P)}{\sin \frac{1}{2}(Z - P)} \tan \tfrac{1}{2}(z - p),$$

$$\log \tan \tfrac{1}{2}m = \log \sin \tfrac{1}{2}(Z + P) + \operatorname{colog} \sin \tfrac{1}{2}(Z - P) + \log \tan \tfrac{1}{2}(z - p).$$

$Z + P$	$168^\circ\,39'$
$Z - P$	$86^\circ\ \ 9'$
$z - p$	$46^\circ\,11'$
$\frac{1}{2}(Z + P)$	$84^\circ\,19.5'$
$\frac{1}{2}(Z - P)$	$43^\circ\ \ 4.5'$
$\frac{1}{2}(z - p)$	$23^\circ\ \ 5.5'$
$\log \sin \frac{1}{2}(Z + P)$	$9.99786 - 10$
$\operatorname{colog} \sin \frac{1}{2}(Z - P)$	0.16561
$\log \tan \frac{1}{2}(z - p)$	$9.62978 - 10$
$\log \tan \frac{1}{2}m$	$9.79325 - 10$
$\frac{1}{2}m$	$31^\circ\,51'$
m	$63^\circ\,42'$

Since m = colat., lat. $= 90^\circ - 63^\circ\,42' = 26^\circ\,18'$ N.

EXERCISES XVII. B

1. An observation taken in New York ($40^\circ\,43'$ N) showed the altitude of the sun to be $52^\circ\,25'$. Its declination was found

* Discarded, since Z and z must terminate in the same quadrant.

to be $12° 15'$. What was the local apparent time of the observation if it was taken in the morning?

2. An afternoon observation at Montreal ($45° 30'$ N) determined the altitude of the sun to be $26° 30'$. Given that the declination of the sun was $8° 0'$ S, find the local apparent time of the observation.
3. Find the altitude and the azimuth of the sun at 3 p.m. in latitude $47° 38'$ N, its declination being $7° 18'$.
4. The declination of a star is $22° 1'$, its hour angle is $15° 8'$. The latitude of the place of observation is $51° 19'$ N. Find the altitude and the azimuth of the star.
5. The declination of a star is $-26° 19'$, its altitude is $31° 5'$, and its azimuth is S $18° 9'$ W. Find the latitude of the observer.
6. The altitude of the sun is $50° 32'$, its declination is $12° 38'$, its azimuth S $12° 6'$ W. Find the latitude and the local apparent time.
7. Find the local apparent time of sunset in Chicago ($41° 50'$ N) on a day when the declination of the sun is $-7° 30'$.

   Suggestion. At sunset the altitude of the sun is $0°$.

   Note. In practice a correction must be made in problems of this type for the refraction of the rays of the sun by the atmosphere of the earth. Another correction must be made for the angular radius of the sun.
8. Find the length of the day (sunrise to sunset) in New Orleans ($29° 57'$ N) when the declination of the sun is $-20°$.
9. On the longest day of the year the declination of the sun is $23° 27'$. Find the length of the longest day in latitude (a) $25°$, (b) $45°$, (c) $65°$.
10. On the shortest day of the year the declination of the sun is $-23° 27'$. Find the length of the shortest day in latitude (a) $25°$, (b) $45°$, (c) $65°$.

IMPORTANT FORMULAS

INDEX

ANSWERS

Important Formulas

$$\sin\theta\csc\theta = 1 \qquad \sin^2\theta + \cos^2\theta = 1$$
$$\cos\theta\sec\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$
$$\tan\theta\cot\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$
$$\cot(\theta + \phi) = \frac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta}$$
$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$
$$\cot(\theta - \phi) = \frac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta}$$
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad \cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$
$$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{2}} \qquad \cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$
$$\tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$$
$$\sin\theta + \sin\phi = 2\sin\tfrac{1}{2}(\theta + \phi)\cos\tfrac{1}{2}(\theta - \phi)$$
$$\sin\theta - \sin\phi = 2\cos\tfrac{1}{2}(\theta + \phi)\sin\tfrac{1}{2}(\theta - \phi)$$

$$\cos\theta + \cos\phi = 2\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)$$
$$\cos\theta - \cos\phi = -2\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)$$

Plane triangles.

Law of sines: $\dfrac{a}{\sin A} - \dfrac{\quad}{\sin B} \quad \dfrac{c}{\sin}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$ *

Law of tangents: $\dfrac{a-b}{a+b} = \dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$ *

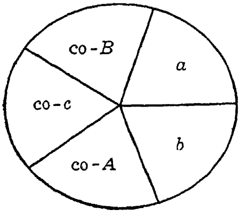
$\tan\frac{1}{2}A = \quad - a$ $\qquad s = \frac{1}{2}(a+b+c),$

$$\sqrt{(s-a)(s-b)(s-c)}$$

Spherical triangles.

Napier's rules (for right triangles):

$$\sin(\text{mid. part}) = \tan(\text{adj. part})\tan(\text{adj. part})$$
$$= \cos(\text{opp. part})\cos(\text{opp. part})$$



Law of sines: $\dfrac{\sin a}{\sin A} - \dfrac{\sin b}{\sin B} \quad \dfrac{\sin c}{\sin C}$

Law of cosines for sides:

$$\cos a = \cos b\cos c + \sin b\sin c\cos A \text{ *}$$

Law of cosines for angles:

$$\cos A = -\cos B\cos C + \sin B\sin C\cos a \text{ *}$$

Napier's analogies:

$$\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c} \text{ *}$$

$$\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c} \text{ *}$$

* Two other formulas may be obtained by changing the letters.

$$\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C} \; *$$

$$\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(A+B)}{\cot \frac{1}{2}C} \; *$$

$$\tan \tfrac{1}{2}A = \frac{\tan r}{\sin(s-a)} \; * \qquad s = \tfrac{1}{2}(a+b+c),$$

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\tan \tfrac{1}{2}a = \tan R \cos(S-A), * \qquad S = \tfrac{1}{2}(A+B+C),$$

$$\tan R = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}$$

* Two other formulas may be obtained by changing the letters.

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(Numbers refer to pages)

Answers to Odd-Numbered Exercises

Exercises I. A and B, page 5

	sin A cos B	cos A sin B	tan A cot B	csc A sec B	sec A csc B	cot A tan B
1.	$\frac{4}{5}$			$\frac{5}{4}$	$\frac{5}{3}$	$\frac{3}{4}$
3.	$\frac{2\sqrt{13}}{13}$	$\frac{3\sqrt{13}}{13}$	$\frac{2}{3}$	$\frac{\sqrt{13}}{2}$	$\frac{\sqrt{13}}{3}$	$\frac{3}{2}$
5.	$\frac{2}{3}$	$\frac{\sqrt{5}}{3}$	$\frac{2\sqrt{5}}{5}$	$\frac{3}{2}$	$\frac{3\sqrt{5}}{5}$	$\frac{\sqrt{5}}{2}$
7.	$\frac{8}{17}$	$\frac{15}{17}$	$\frac{8}{15}$	$\frac{17}{8}$	$\frac{17}{15}$	$\frac{15}{8}$
9.	$\frac{7}{25}$		$\frac{7}{24}$		$\frac{25}{24}$	$\frac{24}{7}$
11.	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
13.	$\frac{3\sqrt{10}}{10}$	$\frac{\sqrt{10}}{10}$	3	$\frac{\sqrt{10}}{3}$	$\sqrt{10}$	$\frac{1}{3}$

15. $\frac{13}{15}$. **17.** $\frac{1}{48}$. **19.** (a) $\frac{3}{4}, \frac{\sqrt{7}}{4}, \frac{3\sqrt{7}}{7}$; (b) $\frac{\sqrt{7}}{4}$ $\frac{\sqrt{7}}{3}$.

Exercises I. C, page 8

	sin A	cos A	tan A	csc A	sec A	cot A
1.	$\frac{3}{5}$		$\frac{3}{4}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{4}{3}$
3.	$\frac{5\sqrt{26}}{26}$	$\frac{\sqrt{26}}{26}$	5	$\frac{\sqrt{26}}{5}$	$\sqrt{26}$	
5.	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$		
7.		$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
9.	$\frac{2\sqrt{29}}{29}$	$\frac{5\sqrt{29}}{29}$		$\frac{\sqrt{29}}{2}$	$\sqrt{29}$	$\frac{5}{2}$

	sin A	cos A	tan A	csc A	sec A	cot A
11.	$\frac{2\sqrt{29}}{29}$	$\frac{5\sqrt{29}}{29}$	$\frac{2}{5}$	$\frac{\sqrt{29}}{2}$	$\frac{\sqrt{29}}{5}$	
13.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$		$\frac{\sqrt{3}}{3}$
15.	$\frac{\sqrt{5}}{5}$	$\frac{2\sqrt{5}}{5}$		$\sqrt{5}$	$\frac{\sqrt{5}}{2}$	2
17.		$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
19.	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$		2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
21.		$3\sqrt{5}$	$\frac{2\sqrt{5}}{15}$	7	$\frac{7\sqrt{5}}{15}$	$\frac{3\sqrt{5}}{2}$

23. $\cos A \quad \frac{m^2 -}{m^2 + n^2}$, $\tan A \quad \frac{2mn}{m^2 - n^2}$, $\csc A \quad \frac{+ n^2}{2mn}$.

$$\sec A = \frac{m^2 + n^2}{m^2 - n^2}, \qquad \cot A = \frac{m^2 - n^2}{2mn}.$$

Exercises I. D, page 11

1. 0.8802. **3.** 0.2805. **5.** 0.7112. **7.** 0.0029. **9.** 343.77.
11. 36° 40′. **13.** 17° 0′. **15.** 68° 30′. **17.** 8° 20′. **19.** 77° 10′.
21. 24° 0′. **23.** 0.8420. No.

Exercises II. A, page 19

1. $B = 55°$, $a = 2.87$, $b = 4.10$.
3. $B = 53°$, $a = 39.94$, $c = 66.37$.
5. $A = 53° 30'$, $B = 36° 30'$, $c = 28.60$.
7. $A = 72° 30'$, $a = 293.1$, $c = 307.3$.
9. $A = 16° 40'$, $B = 73° 20'$, $c = 0.8937$. **11.** 37.3 ft., 38.6 ft.
13. 46°. **15.** 63.1 ft. **17.** 1418 ft. **19.** 120.6 ft.

Exercises II. B, page 23

1. 0.5185. **3.** 0.8887. **5.** 0.8200. **7.** 0.3528. **9.** 0.7001.
11. 0.0026. **13.** 49.923. **15.** 0.4603. **17.** 21° 18′. **19.** 21° 19′
21. 19° 12′. **23.** 67° 46′. **25.** 0° 45′. **27.** 6° 5′. **29.** 11° 28′
31. $A = 20°$, $B = 70°$, $b = 18.79$.
33. $B = 32° 48'$, $a = 0.0240$, $b = 0.0155$.
35. $A = 29° 49'$, $B = 60° 11'$, $b = 32.27$.
37. $B = 70° 16'$, $b = 63.56$, $c = 67.54$.
39. $B = 44° 58'$, $a = 8.230$, $c = 11.63$.
41. $A = 7° 22'$, $B = 82° 38'$, $b = 1.825$.

43. $B = 78° 59'$, $a = 19.42$, $b = 99.73$.
45. $A = 7° 4'$, $B = 82° 56'$, $b = 99.54$.
47. 161.4 ft., 32° 36′, 57° 24′. **49.** 80.87 ft. **51.** 130.9 ft.
53. 2.48 ft. **55.** 3.47 ft. **57.** 116.1 ft.

Exercises II. C, page 28

1. 14.2 knots, S 28° 12′ W. **3.** 24.2 ft./sec., 65° 34′.
5. 53° 8′ with upstream direction; (b) 15 min.
7. 90° 58′. **9.** 86.04 lb.

Exercises II. D, page 30.

1. 99° 30′, 9.83 in., 47.6 sq. in. **3.** 21° 58′, 79° 1′, 79° 1′.
5. 122° 6′. **7.** 8.42 in. **9.** 41° 25′, 198.4 sq. ft.
11. (a) 16.18 in., 15.39 in., 769.4 sq. in.; (b) 21.93 in., 20.61 in., 1391 sq. in.; (c) 21.60 in., 21.33 in., 1442 sq. in.
13. 15.35 ft., 12.42 ft.

Exercises II. E, page 34

1. $C = 70°$, $b = 29.5$, $c = 28.2$. **3.** $B = 74° 2'$, $C = 35° 58'$, $b = 8.2$.
5. $A = 95° 44'$, $B = 40° 27'$, $C = 43° 48'$.
7. $A = 50° 16'$, $B = 29° 44'$, $b = 52.9$.
9. 0.13 mi. = 686 ft. **11.** 127 ft. **13.** 105 ft. **15.** 409 ft.

Exercises III. A, page 39

1. 12.3, 29.9, 4.1, 1.40, 0.25, 0.22, 68, 63.2, 2.000, 2.000, 2.36, 2.34, 2.35, 2.35.
3. 0.002, 0.00005, 0.00001, 0.25, 0.02.
5. $1\underline{0}.\underline{0}2$, $1\underline{0}.2\underline{0}$, $0.2\underline{0}$, 0.02, $0.02\underline{0}$, $25\underset{?}{0}\underset{?}{0}\underset{?}{0}$ $25\underline{0}6$, $0.003\underline{0}\underline{0}$, $0.2\underline{0}5\underline{0}\underline{0}$, $2\underline{0}5\underset{?}{0}\underset{?}{0}$.
7. 18,000,000, 0.000,023,5, 848,200,000, 0.000,000,003,7.

Exercises III. B, page 43

1. 1490. **3.** 55.04. **5.** 231700. **7.** 18800. **9.** 1,242,800.
11. 2.93. **13.** 27.95. **15.** 147.2. **17.** 190500. **19.** 2.60.
21. 41.02. **23.** 4.241. **25.** 0.8272.

Exercises IV. A, page 48

1. 2. **3.** 3. **5.** −1. **7.** −1. **9.** −3. **11.** −1.
13. 1. **15.** 3. **17.** 0. **19.** 5. **21.** −2. **23.** 1.
25. 1. **27.** 3. **29.** −1. **31.** −2. **33.** 7. **35.** −1.

Exercises IV. B, page 50

1. 1.83251. **3.** 2.55509. **5.** 0.30103. **7.** 3.69897.
9. 3.92572. **11.** 8.33365 − 10. **13.** 5.39794. **15.** 0.89492.
17. 1.20276. **19.** 0.47195. **21.** 3.83154. **23.** 4.73501.
25. 0.80023. **27.** 6.94298 − 10. **29.** 0.99992. **31.** 4.9
33. 6.00004 − 10. **35.** 2.91908.

Exercises IV. C, page 51

1. 5.0000. **3.** 863.00. **5.** 0.64980. **7.** 0.000,000,578,80.
9. 0.069890. **11.** 0.049074. **13.** 0.001,576,4. **15.** 0.066567.
17. 1,427,700. **19.** 6.8305. **21.** 88.202. **23.** 10.002.

Exercises IV. D, page 56

1. 1489. **3.** 1.16. **5.** 15700. **7.** 1217. **9.** 0.2247.
11. 5.117. **13.** 0.9564. **15.** 92,024,000. **17.** 0.62764. **19.** 7.2292.
21. 38,122,000,000,000. **23.** 299.83. **25.** 0.97422. **27.** 0.4544.
29. 47.002. **31.** 1.146×10^{14}. **33.** 2.1064. **35.** 2.7314.
37. 2.9295. **39.** −0.020629. **41.** −21.544. **43.** 19.594.

Exercises IV. E, page 59

In exercises 1–23, −10 is to be appended.

1. 9.68557. **3.** 9.99067. **5.** 10.50704. **7.** 9.34276.
9. 9.81519. **11.** 9.13078. **13.** 10.23101. **15.** 9.84933.
17. 9.71647. **19.** 9.22613. **21.** 9.92504. **23.** 10.71142.
25. $20^\circ 14'$. **27.** $63^\circ 41'$. **29.** $57^\circ 0.5'$. **31.** $11^\circ 0.1'$. **33.** $57^\circ 37.8'$.
35. $38^\circ 12.4'$. **37.** $39^\circ 11.8'$. **39.** $81^\circ 13.5'$. **41.** $49^\circ 25.5'$. **43.** $88^\circ 24.4'$.
45. $87^\circ 15.0'$. **47.** Impossible. **49.** 2.855. **51.** 97.035. **53.** 0.18058.
55. 147.33. **57.** 0.86142. **59.** 1362.4. **61.** $37^\circ 52.9'$.

Exercises V. A, page 63

1. $A = 39^\circ 25'$, $B = 50^\circ 35'$, $c = 1250$; 383100.
3. $A = 47^\circ 53'$, $B = 42^\circ 7'$, $b = 0.1846$; 0.01885.
5. $A = 51^\circ 52'$, $B = 38^\circ 8'$, $a = 6385$; 16,000,000.
7. $A = 31^\circ 45'$, $b = 77.63$, $c = 91.29$; 1865.
9. $A = 66^\circ 51'$, $a = 1765$, $c = 1920$; 666200.
11. $A = 26^\circ 23.0'$, $B = 63^\circ 37.0'$, $b = 5728.8$; 8,139,400.
13. $A = 33^\circ 39.4'$, $B = 56^\circ 20.6'$, $a = 574.16$; 247560.
15. $A = 63^\circ 42.8'$, $b = 165.90$, $c = 374.61$; 27861.
17. $A = 37^\circ 50.2'$, $a = 44.909$, $b = 57.820$; 1298.3.
19. (a) 101.05; (b) 7319.2. **21.** 12.478 cm.

Exercises VI. A, page 70

		cos	tan		sec	cot
1.	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
3.	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
5.	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	
7.	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$

9. $\frac{3}{2} + \frac{\sqrt{3}}{2}$. **11.** $-\frac{1}{2} + 5\sqrt{3}$. **13.** $-3 \quad \frac{2\sqrt{3}}{3}$. **15.** $-\frac{13}{4} + \sqrt{3}$.

17. $\frac{3}{2} - \sqrt{2}$. **19.** $2 + \frac{4\sqrt{3}}{3}$. **21.** 4. **23.** 4. **25.** $\frac{3}{5}$. **27.** 0.

Exercises VI. B, page 78

1. (a) $\sin 20°$ or $\cos 70°$; (b) $-\cos 35°$ or $-\sin 55°$;
(c) $-\tan 80°$ or $-\cot 10°$; (d) $\csc 50°$ or $\sec 40°$;
(e) $-\sec 8°$ or $-\csc 82°$; (f) $-\cot 82°$ or $-\tan 8°$;
(g) $\sin 43°$ or $\cos 47°$; (h) $-\cos 84° 50'$ or $-\sin 5° 10'$;
(i) $-\tan 17° 56'$ or $-\cot 72° 4'$; (j) $-\cot 54° 42'$ or $-\tan 35° 18'$;
(k) $\sin 65° 39'$ or $\cos 24° 21'$; (l) $-\cos 87° 47.2'$ or $-\sin 2° 12.8'$.

3. (a) 0.57358; (b) -0.40674; (c) -3.7321; (d) 1.5617;
(e) 0.77715; (f) -0.97499; (g) -0.60626; (h) 0.97622;
(i) -0.29654; (j) 0.30486; (k) -0.36397; (l) 0.09277.

5. 0.

7. (a) 18° or 162°; (b) 60° 10′; (c) 70° 50′; (d) 30° 20′;
(e) 42° 10′ or 137° 50′; (f) 140° 30′.

Exercises VII. A, page 83

1. $C = 30°$, $b = 12.6$, $c = 6.4$. **3.** $B = 37° 10'$, $a = 3.5$, $c = 4.1$.
5. $A = 93° 40'$, $a = 324$, $c = 314$. **7.** 9.4, 6.7. **9.** 12.6, 5.34.
11. 92.2 ft. **13.** 110 ft.

Exercises VII. B, page 87

1. $B = 23° 41'$, $C = 116° 19'$, $c = 11.2$.
3. $A = 23° 48'$, $C = 120° 2'$, $c = 45.5$.
5. $B = 43° 37'$, $C = 63° 3'$, $c = 2.3$.
7. $A = 84° 12'$, $B = 80° 8'$, $b = 34.7$;
$A' = 95° 48'$, $B' = 68° 32'$, $b' = 32.7$.
9. 7.48 in., 8.03 in. **11.** 54.3 ft.

Exercises VII. C, page 90

1. $A = 51°$, $C = 69°$, $b = 5.6$. **3.** $B = 41°$, $C = 121°$, $a = 0.77$.
5. $A = 53° 25'$, $B = 31° 35'$, $c = 285$. **7.** 14.4 mi. **9.** 3.62 in., 7.20 in.
11. 175 yd.

Exercises VII. D, page 91

1. $A = 28° 57'$, $B = 46° 34'$, $C = 104° 29'$.
3. $A = 75° 26'$, $B = 56° 4'$, $C = 48° 30'$.
5. $A = 16° 16'$, $B = 73° 44'$, $C = 90° 0'$.
7. $A = 38° 56'$, $B = 34° 11'$, $C = 106° 54'$.
9. 35° 42′ E or W of S. **11.** 57° 10′, 122° 50′, 23.5 in. **13.** 12.07.

Exercises VII. E, page 94

1. $A = 33° 9.9'$, $a = 435.71$, $c = 787.53$; 156030.
3. $B = 15° 57.0'$, $b = 5.4420$, $c = 17.865$; 36.400.
5. $B = 111° 11.3'$, $a = 102.19$, $b = 491.06$; 21190.
7. $B = 42° 12.8'$, $a = 514.73$, $c = 1025.0$; 177250.
9. $A = 42° 7.7'$, $a = 0.18940$, $c = 0.26964$; 0.013004.
11. 15.223 in., 18.439 in.

Exercises VII. F, page 95

1. $A = 57° 59.9'$, $C = 23° 36.6'$, $c = 29.526$; 913.08.
3. $A = 104° 32.3'$, $B = 40° 1.9'$, $a = 5888.4$; 6,678,200;
$A' = 4° 36.1'$, $B' = 139° 58.1'$, $a' = 488.04$; 553500.
5. $A = 63° 8.3'$, $B = 67° 32.8'$, $b = 89.534$; 2933.9;
$A' = 116° 51.7'$, $B' = 13° 49.4'$, $b' = 23.147$; 758.48.
7. $A = 103° 21.9'$, $C = 48° 48.8'$, $a = 0.67733$; 0.082812;
$A' = 20° 59.5'$, $C' = 131° 11.2'$, $a' = 0.24939$; 0.030491.
9. $A = 134° 37.3'$, $C = 25° 8.2'$, $a = 94.370$; 919.44;
$A' = 4° 53.7'$, $C' = 154° 51.8'$, $a' = 11.314$; 110.23.
11. No solution. **13.** 7423 ft. or 3344 ft.

Exercises VII. G, page 99

The answer for the third side may differ slightly from that given; it depends on the formula used.

1. $A = 57° 50'$, $B = 58° 32'$, $c = 300.9$; 36490.
3. $A = 38° 52.7'$, $B = 8° 49.0'$, $c = 43.017$; 120.36.
5. $A = 153° 17.5'$, $C = 14° 14.0'$, $b = 32.381$; 268.22.
7. $A = 23° 26.2'$, $C = 19° 2.6'$, $b = 819.00$; 64450.
9. $B = 46° 23.8'$, $C = 90°$, $a = 17120$; 153,880,000.
11. 2577 ft.

Exercises VII. H, page 103

1. $A = 44° 4.8', B = 101° 44.4', C = 34° 10.8'$; 6212.4.
3. $A = 30° 41.8'$, $B = 99° 25.2'$, $C = 49° 53.2'$; 74.745.
5. $A = 33° 32.6'$, $B = 50° 40.8'$, $C = 95° 46.6'$; 1,742,200,000.
7. $A = 53° 34.0'$, $B = 26° 5.0'$, $C = 100° 21.0'$; 483.07.
9. $A = 28° 11.8'$, $B = 34° 4.8'$, $C = 117° 43.2'$; 1.8836.
11. 41.51 ft.

Exercises VII. I, page 105

1. $C = 52° 15.9'$, $b = 621.94$, $c = 516.16$; 132100.
3. $A = 65° 21.8'$, $b = 1.6389$, $c = 4.7821$; 3.5621.
5. $A = 127° 9.4'$, $B = 6° 24.4'$, $C = 46° 26.2'$; 0.027977.
7. $A = 27° 28.0'$, $B = 125° 55.4'$, $c = 265.29$; 29345.
9. $A = 46° 26.3'$, $B = 6° 24.4'$, $b = 74260$; 279,762,000.
11. $B = 81° 12.2'$, $a = 303.45$, $c = 271.32$; 40682.
13. $A = 46° 23.8'$, $C = 29° 21.2'$, $b = 9.8396$; 17.730.
15. $A = 26° 21.6'$, $B = 106° 40.6'$, $C = 46° 57.8'$; 788.70.
17. $C = 33° 43.0'$, $a = 487.51$, $b = 689.63$; 93310.
19. $A = 99° 40.1'$, $B = 28° 20.0'$, $c = 182.37$; 9873.5.
21. 975.25 ft. **23.** N 80° 2′ W, S 19° 6′ E. **25.** 885.2 ft.
27. 31830 ft. **29.** 927.0 ft., 742.6 ft., 35° 26.5′.
31. 751.5 ft. **33.** 39° 41′. **35.** 42.9 ft. **37.** 19.806, 35.690, 44.504.
39. 57.67 rd., 96.11 rd., 134.56 rd. **49.** 48° 26′.

Exercises VII. J, page 112

1. 15.18 lb., 44° 24′.
3. 30° with vertical and from front to back of windows.
5. 49° 28′. **7.** 36.5 mi./hr., N 18° 21′ W.
9. 127° 10′, 90° 22′, 142° 27′.

Exercises VIII. A, page 117

	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
1.		$\frac{5}{13}$	$\frac{12}{5}$	$\frac{13}{12}$	$\frac{13}{5}$	$\frac{5}{12}$
3.	$-\frac{2\sqrt{13}}{13}$	$\frac{3\sqrt{13}}{13}$		$-\frac{\sqrt{13}}{2}$	$\frac{\sqrt{13}}{3}$	$-\frac{3}{2}$
5.	$\frac{\sqrt{21}}{5}$		$-\frac{\sqrt{21}}{2}$	$\frac{5\sqrt{21}}{21}$	$-\frac{5}{2}$	$-\frac{2\sqrt{21}}{21}$
7.	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$		-1
9.	$-\frac{7}{25}$	$-\frac{24}{25}$		$-\frac{25}{7}$	$-\frac{25}{24}$	$\frac{24}{7}$

11.		$\pm\frac{\sqrt{3}}{2}$	$\sqrt{3}$	2	$\pm\frac{2\sqrt{3}}{3}$	$\pm\sqrt{3}$
13.	$\pm\frac{2\sqrt{29}}{29}$	$\mp\frac{5\sqrt{29}}{29}$		$\pm\frac{\sqrt{29}}{2}$	$\mp\frac{\sqrt{29}}{5}$	$-\frac{5}{2}$
15.	$\pm\frac{2\sqrt{29}}{29}$	$\mp\frac{5\sqrt{29}}{29}$	$\frac{2}{5}$	$\pm\frac{\sqrt{29}}{2}$	$\pm\frac{\sqrt{29}}{5}$	
17.	$\pm\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\mp\sqrt{3}$	$\pm\frac{2\sqrt{3}}{3}$		$\mp\frac{\sqrt{3}}{3}$
19.	$\pm\frac{\sqrt{5}}{5}$	$\pm\frac{2\sqrt{5}}{5}$		$\pm\sqrt{5}$	$\pm\frac{\sqrt{5}}{2}$	2
21.	$\frac{1}{3}$	$\pm 2\sqrt{2}$	$\sqrt{2}$		$\pm\frac{3\sqrt{2}}{4}$	$\pm 2\sqrt{2}$
23.	$\pm\frac{\sqrt{3}}{2}$	$\mp\frac{1}{2}$		$\pm\frac{2\sqrt{3}}{3}$	∓ 2	$-\frac{\sqrt{3}}{3}$
25.	$\pm\frac{2\sqrt{2}}{3}$		$\mp 2\sqrt{2}$	$\pm\frac{3\sqrt{2}}{4}$	-3	$\pm\sqrt{2}$
27.	$\pm\frac{10\sqrt{101}}{101}$	$\mp\frac{\sqrt{101}}{101}$	10	$\pm\frac{\sqrt{101}}{10}$	$\pm\sqrt{101}$	
29.	$\pm\frac{\sqrt{6}}{3}$	$\pm\frac{\sqrt{3}}{3}$		$\pm\frac{\sqrt{6}}{2}$	$\pm\sqrt{3}$	$\frac{\sqrt{2}}{2}$

31. (a) $\pm\frac{33}{40}$, $\pm\frac{29}{120}$; (b) $\pm\frac{6}{4}$ (c) $\frac{199}{85}$, [illegible]; (d) $\pm\frac{5}{9}$;
(e) $\pm\frac{527}{56}$, $\pm\frac{289}{56}$; (f) $\frac{147}{115}$, $\frac{3}{115}$, $\frac{21}{5}$, $\frac{3}{35}$;
(g) $\frac{4958}{425}$, $\frac{518}{85}$, $\frac{1742}{425}$, $\frac{182}{85}$;
(h) $(192m^2 \pm 416mn + 105n^2)/192$, $(192m^2 \pm 304mn - 105n^2)/192$.

Exercises VIII. B, page 120

41.

$\sin\theta =$	$\sin\theta$	$\pm\sqrt{1-\cos^2\theta}$	$\pm\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\csc\theta}$	$\pm\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\pm\dfrac{1}{\sqrt{1+\cot^2\theta}}$
$\cos\theta =$	$\pm\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\pm\dfrac{1}{\sqrt{1+\tan^2\theta}}$	$\pm\dfrac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\dfrac{1}{\sec\theta}$	$\pm\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$
$\tan\theta =$	$\pm\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\pm\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\pm\dfrac{1}{\sqrt{\csc^2\theta-1}}$	$\pm\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\cot\theta}$
$\csc\theta =$	$\dfrac{1}{\sin\theta}$	$\pm\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\pm\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\csc\theta$	$\pm\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\pm\sqrt{1+\cot^2\theta}$
$\sec\theta =$	$\pm\dfrac{1}{\sqrt{1-\sin^2\theta}}$	$\dfrac{1}{\cos\theta}$	$\pm\sqrt{1+\tan^2\theta}$	$\pm\dfrac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\sec\theta$	$\pm\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$
$\cot\theta =$	$\pm\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\pm\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\pm\sqrt{\csc^2\theta-1}$	$\pm\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\cot\theta$

Exercises VIII. C, page 126

3. $\frac{1}{4}(\sqrt{6} - \sqrt{2}),\ \frac{1}{4}(\sqrt{6} + \sqrt{2}),\ 2 - \sqrt{3},\ 2 + \sqrt{3}$. **9.** $\cos\theta$. **11.** 0.
19. (a) $-\frac{185}{697}$; (b) $\frac{672}{697}$; (c) $-\frac{185}{672}$; (d) $-\frac{672}{185}$; (e) $-\frac{455}{697}$; (f) $\frac{528}{697}$; (g) $-\frac{455}{528}$; (h) $-\frac{528}{455}$.
21. (a) $\pm\frac{171}{221}$; (b) $\pm\frac{140}{221}$; (c) $\frac{171}{140}$; (d) $\frac{140}{171}$; (e) $\pm\frac{21}{221}$; (f) $\pm\frac{220}{21}$; (g) $\frac{21}{220}$; (h) $\frac{220}{21}$.

Exercises VIII. D, page 130

3. $\frac{\sqrt{3}}{2},\ -\frac{1}{2},\ -\sqrt{3},\ -\frac{\sqrt{3}}{3}$
5. $\frac{1}{4}(\sqrt{6} - \sqrt{2}),\ \frac{1}{4}(\sqrt{6} + \sqrt{2}),\ 2 - \sqrt{3},\ 2 + \sqrt{3}$.
7. (a) $\pm\frac{720}{1681}$; (b) $-\frac{1519}{1681}$; (c) $\pm\frac{720}{1519}$; (d) $\pm\frac{1519}{720}$; (e) $\pm\frac{5\sqrt{41}}{41},\ \frac{4\sqrt{41}}{41}$; (f) $\pm\frac{4\sqrt{41}}{41}\ \pm\frac{5\sqrt{41}}{41}$; (g) $\frac{5}{4},\ \frac{4}{5}$; (h) $\frac{4}{5},\ \frac{5}{4}$.

Exercises VIII. E, page 132

1. $2 \sin 30° \cos 10° = \cos 10°$. **3.** $2 \cos 50° \cos 10°$. **5.** $2 \cos 40° \cos 2°$.
7. $2 \sin 32\frac{1}{2}° \cos 7\frac{1}{2}°$. **9.** $2 \sin 50° \cos 18° = 2 \cos 40° \sin 72°$.
11. $2 \sin 47° \cos 3° = 2 \cos 43° \sin 87°$. **13.** $2 \sin 2\theta \cos\theta$.
15. $2 \sin \frac{3}{4}\theta \cos \frac{1}{4}\theta$. **17.** $2 \cos \frac{7}{2}\theta \cos \frac{1}{2}\theta$.

Exercises VIII. F, page 133

23. (a) $\pm\frac{64}{1025},\ \pm\frac{496}{1025}$; (b) $\pm\frac{1023}{1025},\ \pm\frac{897}{1025}$; (c) $\frac{64}{1023},\ \frac{496}{897}$; (d) $\frac{1023}{64},\ \frac{897}{496}$; (e) $\pm\frac{496}{1025},\ \pm\frac{64}{1025}$; (f) $\pm\frac{897}{1025},\ \pm\frac{1023}{1025}$; (g) $\frac{496}{897},\ \frac{64}{1023}$; (h) $\frac{897}{496},\ \frac{1023}{64}$; (i) $\frac{336}{625}$; (j) $\frac{527}{625}$; (k) $\frac{336}{527}$; (l) $\frac{527}{336}$; (m) $\pm\frac{\sqrt{2}}{10},\ \pm\frac{7\sqrt{2}}{10}$; (n) $\pm\frac{7\sqrt{2}}{10},\ \pm\frac{\sqrt{2}}{10}$; (o) $\frac{1}{7},\ -7$; (p) $7,\ -\frac{1}{7}$; (q) $\pm\frac{9\sqrt{82}}{82}$; (r) $\pm\frac{\sqrt{82}}{82}$; (s) ± 9; (t) $\pm\frac{1}{9}$; (u) $\pm\frac{512}{1025},\ \pm\frac{62}{1025}$; (v) $\pm\frac{512}{1025},\ \pm\frac{62}{1025}$; (w) $-\frac{16}{1025},\ -\frac{1}{1}$ (x) $\frac{1984}{1025},\ \frac{16}{1025}$.
27. $\frac{1}{4}\sqrt{10 - 2\sqrt{5}},\ \frac{1}{4}(1 + \sqrt{5}),\ \sqrt{5 - 2\sqrt{5}},\ \frac{1}{5}\sqrt{25 + 10\sqrt{5}}$.
29. $\frac{1}{16}(\sqrt{6} + \sqrt{2})(\sqrt{5} - 1) - \frac{1}{8}(\sqrt{3} - 1)\sqrt{5 + \sqrt{5}}$, $\frac{1}{8}(\sqrt{3} + 1)\sqrt{5 + \sqrt{5}} + \frac{1}{16}(\sqrt{6} - \sqrt{2})(\sqrt{5} - 1)$.
31. 120 ft.

Exercises VIII. G, page 138

1. $\sqrt{2} \sin(\theta - 45°)$. **3.** $13 \cos(\theta + \phi),\ \phi = \operatorname{arccot} \frac{12}{5} = 22° 37'$.
5. $2 \cos(\theta - 60°)$. **7.** $\sqrt{2} \cos(\theta - 45°)$. **9.** $1.2997 \cos(\theta + 73° 44')$.

Exercises IX. A, page 140

1. (a) $\frac{\pi}{18}$; (b) $\frac{7\pi}{36}$; (c) $\frac{4\pi}{15}$; (d) $\frac{7\pi}{18}$; (e) $\frac{5\pi}{6}$; (f) $\frac{14\pi}{9}$; (g) $\frac{\pi}{10}$;
(h) $\frac{20\pi}{9}$; (i) $\frac{7\pi}{120}$; (j) $\frac{11\pi}{80}$; (k) $\frac{641\pi}{240}$; (l) $\frac{13\pi}{135}$.

3. (a) 18°; (b) 15°; (c) 12°; (d) 10°; (e) 120°; (f) 135°; (g) 270°; (h) 150°; (i) 36°; (j) 72°; (k) 108°; (l) 144°; (m) 54°; (n) 84°; (o) 75°; (p) 140°.

5. (a) 28° 38′ 52.4″; (b) 38° 11′ 49.9″; (c) 16° 22′ 12.8″; (d) 162° 20′ 17″; (e) 183° 20′ 47.4″; (f) 70° 49′ 3.3″; (g) 7° 4′ 54.3″; (h) 22° 14′ 52.8″.

7. (a) $\frac{\pi}{3}$; (b) $\frac{5\pi}{6}$; (c) $\frac{\pi}{4}$; (d) 3π

9. (a) $\frac{\pi}{12}$; (b) $\frac{\pi}{720}$; (c) $\frac{5\pi}{18}$; (d) 6π; (e) $\frac{19\pi}{24}$

11. (a) $\frac{\sqrt{3}}{2}$; (b) $-\frac{1}{2}$; (c) 1; (d) $-\sqrt{3}$; (e) $-\sqrt{2}$; (f) 2; (g) -1;
(h) 0.76604; (i) 0.15838; (j) −2.0765; (k) −0.28173;
(l) 0.97095; (m) 0.84147; (n) −0.66628; (o) 1.8856; (p) 2.1520;
(q) 0.01000; (r) 0.86232.

Exercises IX. B, page 144

1. 1.4. **3.** 3 ft. $6\frac{1}{2}$ in. **5.** 10 in. **7.** 1.9263 in. **9.** 2640.
11. (a) $60\pi^{(r)}$/sec.; (b) 240π ft./sec.

Exercises IX. C, page 146

1. 13.5 sq. in., 1.2305 sq. in. **3.** $1\frac{1}{5}^{(r)}$. **5.** 10.05 in.
7. 144 sq. in. **9.** (a) 15 sq. in.; (b) 4.687 cu. in. **11.** 103.0.

Exercises IX. D, page 150

Table IIIa of the Macmillan Logarithmic and Trigonometric Tables was used in obtaining some of these answers.

1. (a) 0.02132; (b) 0.02132; (c) 46.903.
3. (a) 8.19904 − 10; (b) 8.19910 − 10; (c) 1.80090.
5. 153.6. **7.** 2160 mi. **9.** 2.5×10^{13} mi. **11.** 238500 mi.
13. $A = 0°\,45.2'$, $B = 89°\,14.8'$, $c = 57.958$.
15. $A = 174°\,15.4'$, $B = 3°\,3.5'$, $C = 2°\,41.1'$.
17. $A = 59°\,25.0'$, $b = 0.13531$, $c = 0.072393$.

Exercises IX. E, page 152

1. 40. **3.** 2100 ft. **5.** 83 mils. **7.** 43 mils. **9.** 20.
11. 0° 33′ 45″, 2° 48′ 45″, 5° 37′ 30″.

Exercises X. A, page 163

15. $\frac{\pi}{4} + n\pi$.

23(1). 2π. **23(3).** 2π. **23(5).** 4π. **23(7).** 2π. **23(9).** $\frac{\pi}{2}\cdot$
23(11). 4.

Exercises XI. A, page 173

3. $\frac{3\pi}{4}, 2n\pi \pm \frac{3\pi}{4}\cdot$ **5.** $\frac{\pi}{2}, 2n\pi \pm \frac{\pi}{2}\cdot$ **7.** $\frac{\pi}{4}, \frac{\pi}{4} + n\pi$.

9. $-\frac{\pi}{3}, -\frac{\pi}{3} + n\pi$.
11. $0.240, n\pi + (-1)^n\, 0.240$. **13.** $0.980, 0.980 + n\pi$.
15. $1.581, 2n\pi \pm 1.581$. **17.** $0.7297, n\pi + (-1)^n\, 0.7297$.
19. $1.1071, 1.1071 + n\pi$. **21.** $\frac{3}{4}$. **23.** $\frac{9}{13}$. **25.** $-\frac{8}{15}$. **27.** $\pm\frac{20}{29}$. **29.** $\pm\frac{3}{4}$.
31. $-\frac{1}{3}\cdot$ **33.** x. **35.** $\pm\frac{x}{\sqrt{1-x^2}}$ **37.** $\pm\frac{x}{1-x^2}$ **39.** $\pm\frac{x}{\sqrt{1+x^2}}$
41. $\pm\sqrt{1+x^2}$. **45.** $-\frac{528}{697}$ **47.** $1, -\frac{7}{9}$. **49.** $-\frac{1}{9}$. **51.** $\frac{435}{308}$, $-$
53. $\pm\frac{611}{1189}$ **55.** $\pm\frac{24}{25}$ $\pm\frac{2\sqrt{6}}{25}\cdot$ **57.** $\pm\frac{943}{1105}$; $\frac{47}{1105}$, $-\frac{1073}{1105}$, $\pm\frac{817}{1105}$
77. $n\pi + (-1)^n\theta$. **79.** $\theta + n\pi$.

Exercises XII. A, page 181

1. $n \cdot 180°$. **3.** $45° + n \cdot 180°$. **5.** $75°\,58' + n \cdot 180°$.
7. $90° + n \cdot 180°, 210° + n \cdot 360°, 330° + n \cdot 360°$.
9. $90° + n \cdot 180°, 26°\,34' + n \cdot 180°$.
11. $45° + n \cdot 180°, 161°\,34' + n \cdot 180°$.
15. $60° + n \cdot 180°$. **17.** $11\frac{1}{4}° + n \cdot 22\frac{1}{2}°$.
19. $12° + n \cdot 36°$. **21.** $26°\,34' + n \cdot 180°$.
23. $n \cdot 360°, 90° + n \cdot 360°$. **25.** $126°\,13' + n \cdot 360°, 174°\,25' + n \cdot 360°$.
27. $15° + n \cdot 360°, 285° + n \cdot 360°$. **29.** $n \cdot 180° \pm 45°, 90° + n \cdot 180°$.
31. $n \cdot 360°, 45° + n \cdot 90°$. **33.** $n \cdot 360° \pm 50°\,36', n \cdot 360° \pm 129°\,24'$.
35. $n \cdot 180°, 220°\,39' + n \cdot 360°, 319°\,21' + n \cdot 360°$.
37. $240° + n \cdot 360°, 300° + n \cdot 360°$.

39. $x > 0, r = \sqrt{x^2+y^2}, \theta = \text{Arctan}\,\frac{y}{x} + 2n\pi,$

$$-\sqrt{x^2+y^2}, \theta = \pi + \text{Arctan}\,\frac{y}{x} + 2n\pi;$$

$x < 0, r = \sqrt{x^2 + y^2}, \theta = \pi + \text{Arctan}\,\frac{y}{x} + 2n\pi,$

$r = -\sqrt{x^2 + y^2}, \theta = \text{Arctan}\,\frac{y}{x} + 2n\pi;$

$x = 0, y > 0, r = \pm y, \theta = \pm\frac{\pi}{2} + 2n\pi,$

$y < 0, r = \pm y, \theta = \mp\frac{\pi}{2} + 2n\pi,$

$y = 0, r = 0, \theta$ meaningless.

41. $\theta = 45° \, 50' + (-1)^m \cdot 30° \, 20' + (m + 2k) \cdot 180°,$
$\phi = 45° \, 50' - (-1)^m \cdot 30° \, 20' + (m + 2l) \cdot 180°,$
where k, l, m are any integers.

43. $\theta = 50° \, 46' + m \cdot 360°, \quad \phi = 37° \, 46' + n \cdot 360°;$
$\theta = 129° \, 14' + m \cdot 360°, \phi = 217° \, 46' + n \cdot 360°;$
$\theta = 230° \, 46' + m \cdot 360°, \phi = 142° \, 14' + n \cdot 360°;$
$\theta = 309° \, 14' + m \cdot 360°, \phi = 322° \, 14' + n \cdot 360°.$

47. 1.9346. **49.** 0.4797.* **51.** ± 0.8241. **53.** 2.8632.
55. 0, ± 0.9477. **57.** -3.1423.* **59.** Identity. **61.** $n \cdot 180°$.
63. Identity. **65.** Identity.

Exercises XIII. A, page 187

1. $8 + 6i$. **3.** $2 + 5i$. **5.** $6 + 5i$. **7.** $-1 + 7i$. **9.** $1 + 3i$. **11.** 14.
13. $5 - 2i$. **15.** $-5i$. **17.** $11 + 3i$.

Exercises XIII. B, page 189

1. $5\sqrt{2}$ cis 135°. **3.** 2 cis 30°. **5.** 5 cis 306° 52′. **7.** 6 cis 90°.
9. 17 cis 241° 56′. **11.** $\sqrt{13}$ cis 56° 19′. **13.** $\sqrt{26}$ cis 348° 41′.
15. $7\sqrt{2}$ cis 225°. **17.** 10 cis 306° 52′. **19.** $\sqrt{53}$ cis 164° 3′.
21. $\frac{\sqrt{13}}{6}$ cis 33° 41′. **23.** $\frac{5\sqrt{2}}{2} + \frac{5i\sqrt{2}}{2}$. **25.** $-\frac{3\sqrt{2}}{2} \quad \frac{3i\sqrt{2}}{2}$.
27. $10i$. **29.** $-4i$. **31.** $1 - i$. **33.** $8.1915 - 5.7358i$.
35. $-4.6984 - 1.7101i$. **37.** $7.6604 + 6.4279i$.

Exercises XIII. C, page 190

1. 15 cis 110°. **3.** $2\sqrt{2}$ cis 105°. **5.** 12 cis 110°. **7.** 3 cis 90° = $3i$.
9. $\frac{3\sqrt{2}}{2}$ cis 195°.

Exercises XIII. D, page 193

1. 343 cis 54°. **3.** 32 cis 90° = $32i$. **5.** 2500 cis 180° = -2500.
7. cis 176°. **9.** cis 180° = -1.

* Other solutions exist.

11. 10^{-6} cis $300° = 0.000{,}000{,}5(1 - i\sqrt{3})$. **13.** 3 cis 40°, 3 cis 220°.

15. 3 cis 9°, 3 cis 129°, 3 cis 249°.

17. $\sqrt[3]{2}$ cis $20° = 1.1839 + 0.43092i$, $\sqrt[3]{2}$ cis $140°$
$= -0.96514 + 0.80986i$, $\sqrt[3]{2}$ cis $260° = -0.21878 - 1.2408i$.

19. cis $0° = 1$, cis $120° = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$, cis $240° = -\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$.

21. $\sqrt{2}$ cis $45° = 1 + i$, $\sqrt{2}$ cis $105° = -0.36603 + 1.3660i$,
$\sqrt{2}$ cis $165° = -1.3660 + 0.36603i$, $\sqrt{2}$ cis $225° = -1 - i$,
$\sqrt{2}$ cis $285° = 0.36603 - 1.3660i$, $\sqrt{2}$ cis $345° = 1.3660 - 0.36603i$.

23. $\sqrt{2}$ cis $45° = 1 + i$, $\sqrt{2}$ cis $117° = -0.64204 + 1.2601i$,
$\sqrt{2}$ cis $189° = -1.3968 - 0.22123i$, $\sqrt{2}$ cis $261°$
$= -0.22123 - 1.3968i$, $\sqrt{2}$ cis $333° = 1.2601 - 0.64204i$.

25. $1, 0.30902 \pm 0.95106i, -0.80902 \pm 0.58779i$. **27.** $\pm\dfrac{\sqrt{2}}{2}(1 \pm i)$.

29. $\pm(1.8478 + 0.76536i), \pm(0.76536 - 1.8478i)$.

31. Same as Ex. 25, discarding $x = 1$.

Exercises XV. A, page 207

1. $B = 153°\,58.3'$, $a = 67°\,7.0'$, $b = 155°\,46.7'$.

3. $A = 105°\,52.3'$, $a = 117°\,13.7'$, $b = 33°\,32.7'$.

5. $a = 69°\,34.9'$, $b = 134°\,59.4'$, $c = 104°\,16.8'$.

7. $A = 81°\,43.0'$, $a = 70°\,16.2'$, $c = 107°\,58.2'$;
$A' = 98°\,17.0'$, $a' = 109°\,43.8'$, $c' = 72°\,1.8'$.

9. $A = 78°\,31.9'$, $b = 112°\,48.5'$, $c = 94°\,46.8'$.

11. $A = 127°\,23.3'$, $B = 109°\,52.2'$, $b = 115°\,19.6'$.

13. $A = 74°\,15.2'$, $B = 30°\,30.8'$, $a = 57°\,41.5'$.

15. No solution.

17. $B = 72°\,54.2'$, $b = 46°\,29.5'$, $c = 49°\,21.5'$;
$B' = 107°\,5.8'$, $b' = 133°\,30.5'$, $c' = 130°\,38.5'$.

19. $B = 20°\,49.8'$, $a = 44°\,44.0'$, $c = 46°\,40.1'$.

21. $\arctan\sqrt{2} = 54°\,44'$.

Exercises XV. B, page 208

1. $A = 64°\,40.4'$, $B = 49°\,47.1'$, $C = 106°\,2.0'$.

3. $B = 111°\,25.9'$, $a = 117°\,4.3'$, $b = 108°\,59.2'$.

5. $B = 28°\,14.0'$, $C = 78°\,53.3'$, $b = 28°\,49.4'$;
$B' = 151°\,46.0'$, $C' = 101°\,6.7'$, $b' = 151°\,10.6'$.

7. $A = 118°\,32.6'$, $B = 33°\,20.4'$, $C = 66°\,28.3'$.

9. $A = 47°\,25.6'$, $C = 107°\,50.2'$, $a = 50°\,40.8'$;
$A' = 132°\,34.4'$, $C' = 72°\,9.8'$, $a' = 129°\,19.2'$.

Exercises XV. C, page 209

1. $B = 100° 14.4'$, $a = c = 71° 19.9'$.
3. $A = C = 103° 28.4'$, $b = 110° 37.6'$.
5. $B = C = 49° 1.3'$, $b = c = 78° 20.3'$;
$B' = C' = 130° 58.7'$, $b' = c' = 101° 39.7'$.
7. $a = b = 94° 16.1'$, $c = 99° 48.2'$.
9. $B = 119° 35.4'$, $C = 62° 1.5'$, $b = 110° 32.6'$.
11. $A = B = C = 60° 15.2'$. **13.** $A = B = C = 102° 7.8'$.
15. $a = b = c = 98° 30.5'$.

Exercises XVI. A, page 220

1. (a) Obtuse; (b) acute; (c) acute. **3.** Obtuse. **5.** a obtuse, c acute.
7. Acute: A; obtuse: $\frac{1}{2}(A + C)$, $+ C)$, B, C; $90°$: $\frac{1}{2}(A + B)$.

Exercises XVI. B, page 223

1. $A = 128° 4.2'$, $B = 51° 34.2'$, $C = 73° 14.6'$.
3. $A = 65° 10.0'$, $B = 98° 50.6'$, $C = 125° 17.8'$.
5. $A = 77° 36.0'$, $B = 63° 17.0'$, $C = 107° 23.2'$.
7. $a = 47° 44.8'$, $b = 132° 40.6'$, $c = 103° 11.6'$.
9. No solution.
11. $A = 45° 25.0'$, $B = 33° 59.4'$, $C = 118° 42.0'$.
13. $a = 83° 5.8'$, $b = 102° 31.6'$, $c = 94° 26.2'$.
15. No solution.
17. $a = 126° 36.6'$, $b = 118° 13.4'$, $c = 83° 24.0'$.
19. $a = 46° 11.4'$, $b = 74° 15.4'$, $c = 86° 10.8'$.

Exercises XVI. C, page 227

1. $A = 55° 52.4'$, $B = 20° 10.0'$, $c = 66° 20.8'$.
3. $A = 144° 33.3'$, $B = 112° 46.5'$, $c = 136° 50.8'$.
5. $A = 121° 33.5'$, $B = 43° 13.5'$, $c = 62° 11.6'$.
7. $a = 95° 38.0'$, $b = 41° 52.2'$, $C = 110° 48.8'$.
9. $a = 123° 21.4'$, $c = 84° 15.4'$, $B = 129° 4.6'$.
11. $B = 95° 38.1'$, $C = 97° 26.5'$, $a = 64° 23.2'$.
13. $a = 89° 30.3'$, $c = 62° 32.1'$, $B = 1° 41.4'$.
15. $A = 96° 2.3'$, $B = 125° 43.7'$, $c = 100° 48.0'$.
17. $a = 47° 29.3'$, $b = 50° 6.3'$, $C = 129° 58.6'$.
19. $A = 142° 16.3'$, $B = 46° 7.1'$, $c = 89° 28.2'$.

Exercises XVI. D, page 232

1. $B = 22° 34.8'$, $C = 101° 16.0'$, $c = 50° 36.6'$.
3. $B = 59° 24.4'$, $C = 115° 39.8'$, $c = 97° 33.2'$;

$B' = 120° 35.6'$, $C' = 27° 0.2'$, $c' = 29° 57.4'$.
5. No solution.
7. $C = 101° 42.0'$, $b = 31° 24.7'$, $c = 147° 10.6'$;
$C' = 36° 45.4'$, $b' = 148° 35.3'$, $c' = 19° 20.8'$.
9. No solution.
11. $B = 87° 34.5'$, $C = 53° 6.6'$, $c = 52° 27.2'$;
$B' = 92° 25.5'$, $C' = 25° 26.2'$, $c' = 25° 12.0'$.
13. $B = 97° 21.4'$, $a = 59° 3.2'$, $b = 120° 9.4'$;
$B' = 58° 55.4'$, $a' = 120° 56.8'$, $b' = 48° 19.2'$.
15. $B = 148° 6.3'$, $C = 130° 21.4'$, $c = 62° 9.0'$;
$B' = 31° 53.7'$, $C' = 6° 17.6'$, $c' = 7° 18.4'$.
17. $C = 36° 38.8'$, $b = 51° 17.9'$, $c = 41° 4.6'$.
19. $C = 8° 17.6'$, $b = 125° 23.2'$, $c = 6° 51.2'$;
$C' = 139° 39.0'$, $b' = 54° 36.8'$, $c' = 147° 36.8'$.

Exercises XVI. E, page 233

1. $A = 38° 27.5'$, $B = 92° 38.3'$, $c = 23° 59.0'$.
3. $a = 80° 5.2'$, $b = 70° 10.4'$, $c = 145° 5.0'$.
5. $A = 80° 14.8'$, $b = 145° 55.2'$, $c = 119° 22.6'$.
7. $B = 31° 53.7'$, $C = 6° 17.6'$, $c = 7° 18.4'$;
$B' = 148° 6.3'$, $C' = 130° 21.4'$, $c' = 62° 9.0'$.
9. $A = 98° 56.0'$, $B = 66° 18.0'$, $c = 103° 30.6'$.
11. $a = 98° 44.8'$, $b = 83° 25.0'$, $c = 75° 23.2'$.
13. $a = 74° 36.4'$, $b = 112° 16.6'$, $c = 72° 33.4'$.
15. $C = 36° 38.8'$, $b = 51° 17.9'$, $c = 41° 4.6'$.
17. $A = 50° 30.2'$, $B = 135° 5.5'$, $a = 70° 20.4'$.
19. $A = 53° 30.4'$, $B = 51° 58.4'$, $C = 149° 13.4'$.
21. $B = 85° 41.2'$, $a = 47° 48.4'$, $c = 59° 39.2'$.
23. $A = 23° 17.8'$, $B = 146° 25.6'$, $C = 35° 53.4'$.
25. $C = 53° 30.4'$, $a = 88° 20.8'$, $b = 66° 46.0'$.
27. $C = 139° 39.0'$, $b = 54° 36.8'$, $c = 147° 36.8'$;
$C' = 8° 17.6'$, $b' = 125° 23.2'$, $c' = 6° 51.2'$.
29. $C = 155° 51.0'$, $b = 125° 22.7'$, $c = 155° 48.0'$.
31. 21.67 in., 25.89 sq. in. **33.** 1.645 in.

Exercises XVII. A, page 238

Distances are given in nautical miles. To convert to statute miles, multiply by 1.1516. In Exercises 1–7 the first direction is the bearing of the second point from the first, the second direction is the bearing of the first point from the second.

1. 2229, N 78° 19′ W, N 69° 54′ E. **3.** 6797, S 63° 54′ E, N 55° 32′ W.

5. 5754, S 65° 29′ E, N 51° 19′ W. **7.** 7297, S 15° 34′ E, S 14° 0′ W. **9.** 527 mi. **11.** (a) S 42° 54′ E; (b) S 44° 0′ E. **13.** 190.

Exercises XVII. B, page 245

1. 10:08 a.m. **3.** 34° 30′, S 58° 20′ W. **5.** 30° 13′ N. **7.** 5:33 p.m **9.** (a) $13^h\ 33^m$; (b) $15^h\ 26^m$; (c) $21^h\ 8^m$.